MSRI
Mathematical Sciences Research Institute

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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Iain Gordon

Talk Title: Some Geometry & Combinatorics around the Representations of Cherednik algebras

Date: 01 / 25 / 13 **Time:** 1 : 40 am / pm (circle one)

List 6-12 key words for the talk: symplectic singularities / varieties, Coxeter group, Catalan number, reflection group, Cherednik algebra

Please summarize the lecture in 5 or fewer sentances: Discuss resolving symplectic singularities, complex reflection groups, deformation/quantization of symplectic resolutions. Throughout the discussion, the representation theory of Cherednik algebras was linked to parts of geometry and combinatorics.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Finite group G acting linearly on complex vector space V is generated by reflections / a complex reflection group if it has a generating set $S \subset G$ such that

$$\operatorname{codim}_V \operatorname{Fix}(s) = 1 \quad \forall s \in S.$$

e.g. $\mathfrak{S}_n \circlearrowright \mathbb{C}^n$: $S = \{ (i\, j) : 1 \le i < j \le n \} \rightsquigarrow$ finite Coxeter groups

$\mu_\ell = \{ z \in \mathbb{C} : z^\ell = 1 \} \circlearrowright \mathbb{C}$: $S = \{ z \in \mu_\ell : z \neq 1 \}$

non e.g. $A_5 \circlearrowright V$ never generated by reflections ($A_5 \circlearrowright \Lambda^{\dim V} V$)

There is a classification of irreducible complex reflection groups.

Many constructions in representation theory, geometry and topology, combinatorics single out complex reflection groups naturally.

e.g. $G \circlearrowright V \rightsquigarrow V/G$ - orbit space $\equiv$ affine variety of $\mathbb{C}[V]^G$

V/G is smooth $\iff$ G is generated by reflections on V.

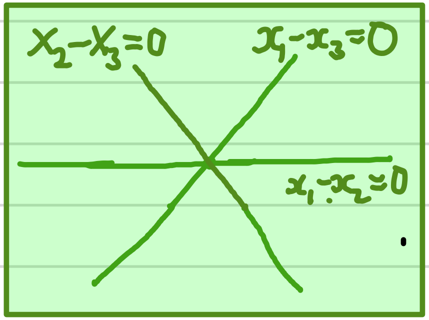
$\iff$ $\mathbb{C}[V]^G$ is a polynomial ring

$\rightsquigarrow \mathbb{C}[V]^G = \mathbb{C}[f_1, \dots, f_n] : \{\deg(f_i)\}_{i=1,\dots,n}$ fundamental degrees of G

e.g. $\mathfrak{S}_n \circlearrowright \mathbb{C}^n$ $\mathbb{C}[\mathbb{C}^n]^{\mathfrak{S}_n}$ = Symmetric polynomials = $\mathbb{C}[e_1, e_2, \dots e_n]$

UNDERSTAND: $G \rightsquigarrow$ hyperplane arrangement $\{H_s = \mathrm{Fix}(s) : s \in S\}$

e.g. S_3



μ_e 

Chen Lie algebra: $\mathfrak{t}(W)$ generated by t_H for H hyperplane, subject to

for each $A \subset H$ is a codimension 2 intersection of hyperplanes

$$\Big[\sum_{A\subset H'} t_{H'}, t_H\Big] = 0$$

$\boxed{\text{universal KZ connection: } d + \sum_H \frac{t_H}{x_H}}$

Each representation V of $\mathfrak{z}(W)$ produces a flat connection on $V^{reg} := V \setminus \bigcup H$.

- quantum groups (Drinfeld-Kohno)
- (Vassiliev) knot invariants
- wonderful models (of de Concini-Procesi)
- Hecke algebras (Broué-Malle-Rouquier)

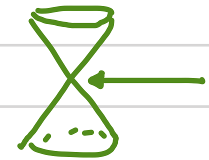
$$c : S/_{\sim} \longrightarrow \mathbb{C} \quad \rightsquigarrow \quad \mathfrak{z}(W) \longrightarrow \mathbb{C}[W]$$

$$t_H \longmapsto \sum_{\substack{s \in S \\ Fix(s) = H}} \frac{c(s)}{1 - \xi_s} s$$

non-trivial eigenvalue of s

$G \curvearrowright T^*V := V \times V^*$ a symplectic vector space.

UNDERSTAND: G-equivariant geometry of T^*V

e.g. $\mu_2 \curvearrowright T^*\mathbb{C} = \mathbb{C}^2$: $\mathbb{C}[\mathbb{C}^2]^{\mu_2} = \mathbb{C}[x^2, xy, y^2] \leftrightsquigarrow$  !

The orbit space $(T^*V)/G$ is not smooth, but mild symplectic singularities.

i.e. its smooth locus it has symplectic structure which extends reasonably everywhere

Classical algebraic geometry may ask

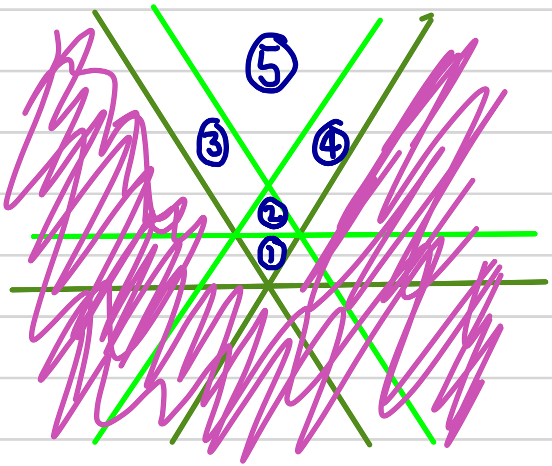
$\exists$ smooth symplectic variety (X,ω) s.t. $\pi : X \to T^*V/G$ is a resolution of singularities which is a SYMPLECTIC ISOMORPHISM over the smooth locus.

$\Longrightarrow$ (Bezrukavnikov–Kaledin) $D^b(\mathbb{C}[T^*V] \rtimes G) \xrightarrow{\sim} D^b(X)$

Classification completed in 2009 by Bellamy e.g. $\mathrm{Hilb}^n\mathbb{C}^2 \to \mathrm{Sym}^n\mathbb{C}^2 = (T^*\mathbb{C}^n)/S_n$

$G \rightsquigarrow \mathrm{Cat}_G(q,t)$ a bigraded Catalan number attached to G

for G a Coxeter group Catalan number counts Shi regions:



$V \setminus \{x_H = 0, 1 : H \text{ hyperplane}\}$

conn. cpts in "dominant region"

$\quad = C_G(1,1)$

e.g. $\mathrm{Cat}_{S_n}(1,1)$: $1, 1, 5, 14, 42, \ldots$

$\rightsquigarrow$ large industry of combinatorics are Macdonald polynomials, Clusters, parking functions, ...

"rational Catalan combinatorics" : $\mathrm{Cat}^{\bullet}_G(q,t)$

Examples have common features:

a) 2-dimensional : • connection ≡ D-module in hyperplane arrangement
• $T^*V = V \times V^*$ in geometry
• q, t in Catalan combinatorics

b) parameters attached to reflections
• Hecke algebra
• stability condition in choice of resolution
• rational Catalan combinatorics $\mathrm{Cat}_c^{\bullet}(q,t)$

!!

Behind such phenomena lurks the

$\boxed{\text{Rational Cherednik Algebra}}$

$G \in V$ irreducible complex reflection group with reflections S

$t \in \mathbb{C}$, $c : S \longrightarrow \mathbb{C}$ constant on conjugacy classes

DEFINITION v1 (PBW style, Etingof-Ginzburg)

$H_{t,c}(G)$ is the quotient of $\mathrm{Tens}(T^*V) \rtimes G$ by relations

$$[x,y] = t\,\omega(x,y) + \sum_{s \in S} c(s)\,\omega_s(x,y)\,s$$

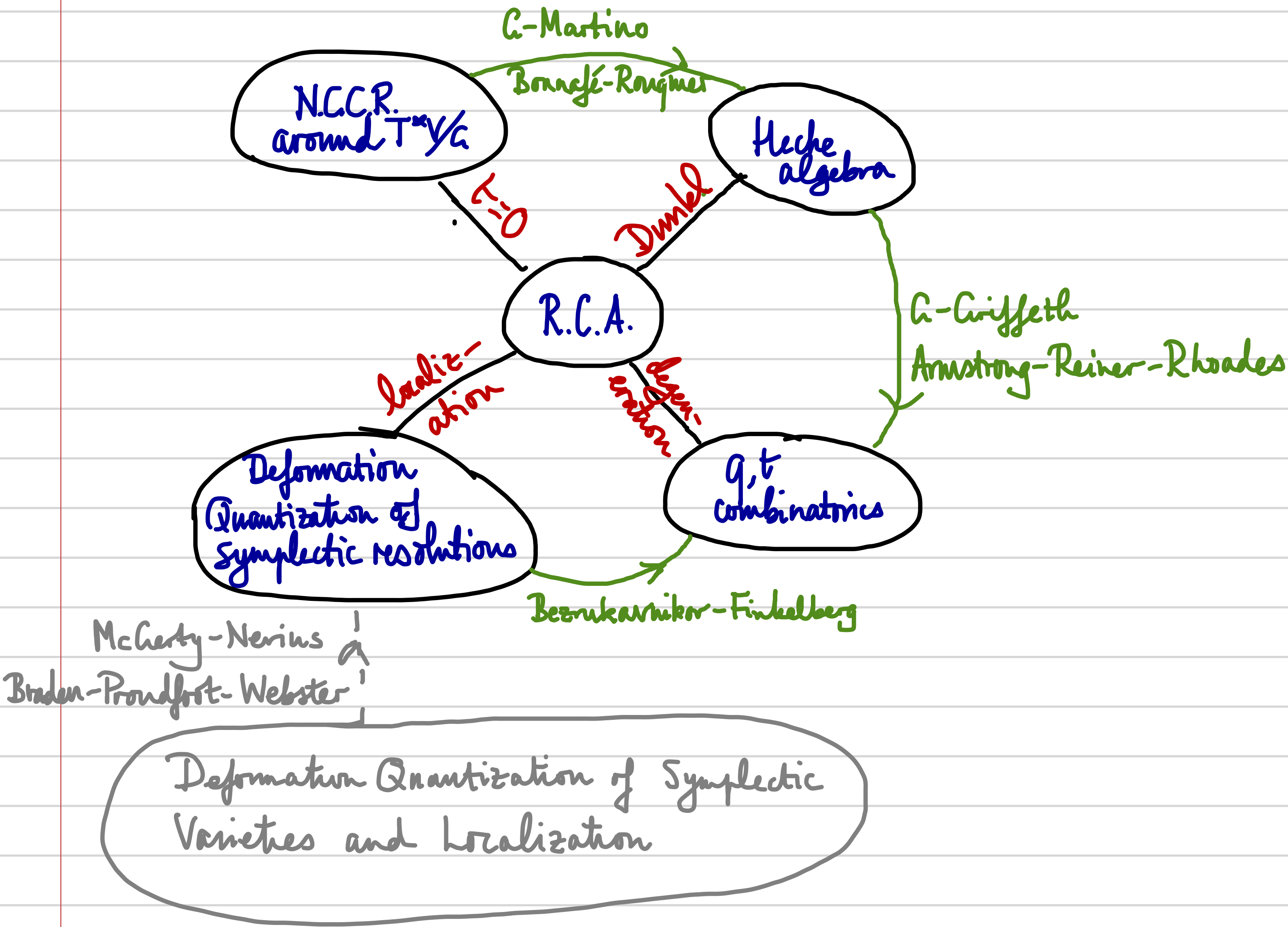
restriction of ω to $\mathrm{Im}(s-1)_{T^*V}$.

In particular: $\mathrm{gr}\, H_{t,c}(G) = \mathbb{C}[T^*V] \rtimes G \longleftarrow$ G-equivariant geometry

DEFINITION v2 (Dunkl embedding)

$H_{1,c}(G)$ is the subalgebra of $D(V^{\mathrm{reg}}) \rtimes G$ generated by

- $\mathbb{C}[V]$
- G
- $\partial_y + \sum_{s \in S} \frac{c(s)}{1-\xi_s} \frac{x_{H_s}(y)}{x_{H_s}} (s-1) \qquad \forall y \in V$


G-Martino
Bonnafé-Rouquier
N.C.C.R. around T^*V/G
Hecke algebra
$t=0$
Dunkl
R.C.A.
G-Griffeth
Armstrong-Reiner-Rhoades
localiz-ation
degen-eration
Deformation Quantization of symplectic resolutions
q,t combinatorics
Bezrukavnikov-Finkelberg
McGerty-Nevins
Braden-Proudfoot-Webster
Deformation Quantization of Symplectic Varieties and Localization

D.Armstrong, V.Reiner & B.Rhoades "Parking Spaces" arXiv:1204.1760

G.Bellamy "Symplectic reflection algebras" arXiv:1210.1239

R.Bezrukavnikov & M.Finkelberg "Wreath Macdonald polynomials and categorical McKay correspondence" arXiv:1208.3696

C.Bonnafe & R.Rouquier "Calogero-Moser versus Kazhdan-Lusztig cells" arXiv:1201.0585

T.Braden, N.Proudfoot & B.Webster "Quantizations of conical symplectic resolutions I: local and global structures" arXiv:1208.3863

K.McGerty & T.Nevins "Derived equivalence for quantum symplectic resolutions" arXiv: 1108.6267

Don't miss Ben Zvi and Chlouveraki next week.