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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Lauren Williams

Talk Title: An Introduction to Cluster Algebras

Date: 01 / 25 / 13 **Time:** 3 : 00 am / pm (circle one)

List 6-12 key words for the talk: quiver, cluster algebra, quiver mutation, seed

Please summarize the lecture in 5 or fewer sentances: Introduce cluster algebras by defining and demonstrating through examples of quiver, seed mutations, cluster and coefficient variables. Indicate conjectures, classification problems and generalization of cluster algebras.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

01/25/13 3:00pm

Lauren Williams: "An Introduction to Cluster Algebras"

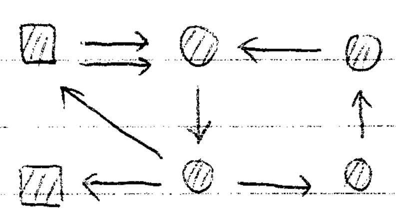
Overview:

+ Cluster algebras are commutative rings with distinguished generators (called cluster variables) having rich combinatorial structure
+ Structure is encoded by a quiver & relations among generators encoded by quiver mutation
+ Introduced by Fomin + Zelevinsky 2000 in the context of Lie Theory, connections to other fields.

Outline:

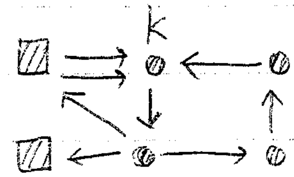
+ What is a cluster algebra?
+ Example
+ Main results + open problems
+ Quantum + non-commutative analogues.

- Quiver Q:
  - finite directed graph
  - multiple edges allowed
  - oriented cycles of length 1 and 2 forbidden
  - 2 types of vertices: "frozen" and "mutable"
  - Ignore edges connecting frozen vertices

Q = 

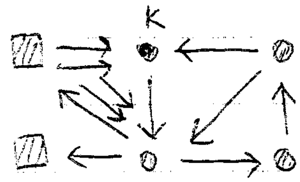
Quiver mutation:

let k be mutable vertex of Q

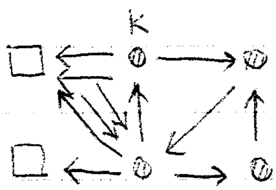


Quiver mutation $\mu_k : Q \to Q'$ is computed by

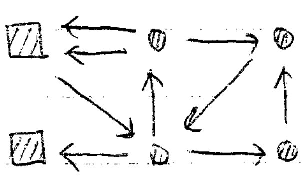
1) For each instance of $j \to k \to \ell$, introduce edge $j \to \ell$



2) Reverse the direction of all edges incident to k



3) Remove oriented 2-cycles

 $= Q'$

Remark: $\mu_k(\mu_k(Q)) = Q$, $\forall k$

Defn: 2 quivers are mutation-equivalence if one can get between via a sequence of mutations.

Seeds:

Defn: Let F be field of rational functions in m independent variables$/\mathbb{C}$.
A seed in F is a pair $(Q, \underline{x})$ consisting of

- Quiver Q on m vertices

- (extended) cluster $\underline{x}$, an m-tuple of alg. indep. (over $\mathbb{Q}$) elements of F indexed by vertices of Q,

where Frozen vertices $\longleftrightarrow$ coefficient variables

Mutable vertices $\longleftrightarrow$ Cluster variables

Cluster $= \{$ cluster variables $\}$

Extended cluster $= \{$ cluster + coeff. variables $\}$

<u>Seed mutation</u>:

Let k be a mutable vertex of Q and let x_k denote a corresp. cluster variable. Then seed mutation

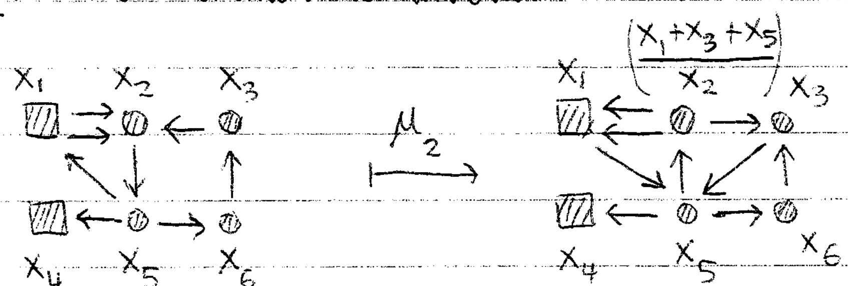
$\mu_k : (Q, \underline{x}) \longmapsto (Q', \underline{x}')$ is defined by

- $Q' = \mu_k(Q)$
- $\underline{x}' = \underline{x} \cup \{x_k'\} \setminus \{x_k\}$, where x_k' is defined by:

$$x_k x_k' = \prod_{\substack{j \leftarrow k \\ \text{in } Q}} x_j + \prod_{\substack{j \rightarrow k \\ \text{in } Q}} x_j$$

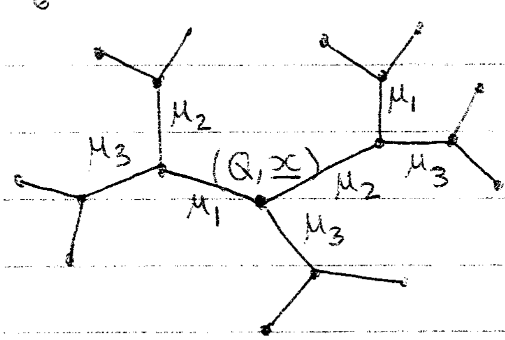
<u>Remark</u>: Seed mutation is involution.

<u>Ex</u>:



seed mutation

$\mathbb{C}(x_1, \ldots, x_6)$



We can keep doing seed mutations to form a tree

Defn.: (of Cluster Algebra)

Let $(Q, \underline{x})$ be a seed in F, where Q has n mutable vertices. Consider n-regular tree $\mathbb{T}_n$ with vertices labeled by seeds, obtained by applying all possible sequences of mutations.

Let $\mathcal{X}$ = union of all cluster + coeff. variables at all nodes of $\mathbb{T}_n$.

The cluster algebra $\mathcal{A} = \mathcal{A}(Q)$ is the subring of F which is generated by $\mathcal{X}$.

n = rank of $\mathcal{A}$

Ex: Consider $(Q, \underline{x}) = \overset{1}{\underset{x_1}{\bullet}} \longrightarrow \overset{2}{\underset{x_2}{\bullet}}$

Let's compute all cluster variables

$$\overset{1}{\underset{x_1}{\bullet}} \longrightarrow \overset{2}{\underset{x_2}{\bullet}} \quad \overset{\mu_1}{\Longmapsto} \quad \overset{1}{\underset{\frac{x_2+1}{x_1}}{\bullet}} \longleftarrow \overset{2}{\underset{x_2}{\bullet}} \quad \overset{\mu_2}{\Longmapsto} \quad \overset{1}{\underset{\frac{x_2+1}{x_1}}{\bullet}} \longrightarrow \overset{2}{\underset{\frac{\frac{x_2+1}{x_1}+1}{x_2}}{\bullet}} = \frac{x_1+x_2+1}{x_1x_2}$$

$$\Big\Downarrow \mu_1$$

$$\overset{1}{\underset{x_2}{\bullet}} \longleftarrow \overset{2}{\underset{x_1}{\bullet}} \quad \overset{\mu_1}{\Longleftarrow} \quad \overset{1}{\underset{\frac{1+x_1}{x_2}}{\bullet}} \longrightarrow \overset{2}{\underset{x_1}{\bullet}} \quad \overset{\mu_2}{\Longleftarrow} \quad \overset{1}{\underset{\frac{1+x_1}{x_2}}{\bullet}} \longleftarrow \overset{2}{\underset{\frac{1+x_1+x_2}{x_1x_2}}{\bullet}}$$

up to relabelling the vertices, we get exactly the same seed as before

↳ The cluster algebra $\mathcal{A}(Q)$ is the subring of $\mathbb{C}(x_1, x_2)$ generated by $\mathcal{X} = \left\{ x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}, \frac{1+x_1}{x_2} \right\}$

Remarks:

1) Each cluster variable is a Laurent polynomial in x_1, x_2
2) Each Laurent poly. has positive coeffs.
3) There are finitely many cluster variables

4) The 2-regular tree closes up to pentagon.

Theorems + Conjectures: (F. + Z.)

let $\mathcal{A} = \mathcal{A}(Q)$ be arbitrary cluster algebra with initial set $(Q, \underline{x})$

Laurent Phen.: Each cluster variable is Laurent poly. in $\underline{x}$.

Positivity Conj.: All coeffs. of Laurent poly above are positive

Finite Type Classification Theorem:

Say $\mathcal{A}$ has finite type if there are finitely many cluster variables

Finite type cluster algebras are classified by Dynkin diagrams

When $\mathcal{A}$ has finite type, $\mathbb{T}_n$ closes up and becomes 1-skeleton of conv. polytope (generalized association)

Generalizations:

+ Quantum cluster algebra (Berenskin + Zelevisky) cluster variables in cluster quasi-commute Laurent phen. still holds (B.Z.)

(Quantum) positivity holds if cluster algebra has a cyclic seed

(Kimana - Qin)

So far: No noncomm. cluster algebra

Quantum rank 2 cluster algebra

$$\underset{Y_1}{\bullet} \rightrightarrows \underset{Y_2}{\bullet}$$

Then F = skew-field of fractions of quantum torus w/ gen. Y_1, Y_2 such that $Y_1 Y_2 = q Y_2 Y_1$

Quantum cluster variables $\leftrightarrow \mathbb{Z}$

$\{Y_m : m \in \mathbb{Z}\}$

$Y_{m-1} Y_{m+1} = q^{r/2} Y_m^r + 1$, clusters are pairs $\{Y_{m-1}, Y_{m+1}\}$ $\forall m$

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