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NOTETAKER CHECKLIST FORM

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Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Michaela Vancliff

Talk Title: The interplay of algebra and geometry in the setting of AS-regular algebras

Date: 01/25/13 **Time:** 10:45 am / pm (circle one)

List 6-12 key words for the talk: AS-regular algebra, gldim, graded Clifford algebra, quadratic form and quadric,

Please summarize the lecture in 5 or fewer sentances: Introduce AS-regular algebras and classifying progress, especially for gldim = 4. Discuss graded skew Clifford algebras (GSCA) and their properties/classification. Using geometric tools and GSCA to exhibit a connection between their algebraic and homological properties and geometric data. Propose open problems in this subject, some are very accessible to junior researchers.

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The Interplay of Algebra and Geometry in the Setting of AS-Regular Algebras

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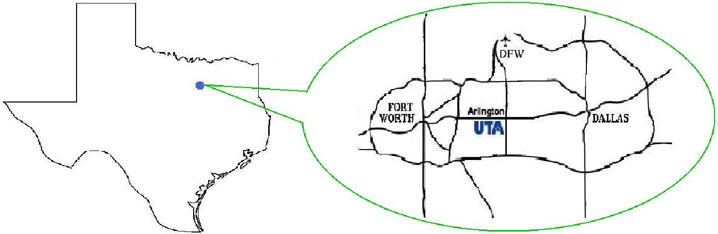

Motivation

Example

Fix a field $\Bbbk$.
Consider the associative $\Bbbk$-algebra, S, on generators $z_1, \ldots, z_n$ with defining relations:

$$z_j z_i = \mu_{ij} z_i z_j, \quad \text{for all } i, \ j,$$

where $0 \neq \mu_{ij} \in \Bbbk$ for $1 \leq i, j \leq n$, $\mu_{ii} = 1$ for all i & $\mu_{ij}\mu_{ji} = 1$ for all i, j.

If all the $\mu_{ij} = 1$, then this algebra is the commutative polynomial ring.

With some $\mu_{ij} \neq 1$, this algebra feels close to commutative, so there *should* be a way to do algebraic geometry with this algebra (& there is).

ATV's idea: use certain modules (representations) in place of points/lines/planes etc....

ATV's Geometry

Henceforth, $\mathbb{k}$ = algebraically closed field.

Definition ([Artin, Tate, Van den Bergh])

Let $A = \bigoplus_{i\geq 0} A_i$, with $A_0 = \mathbb{k}$, be an associative graded $\mathbb{k}$-algebra generated by A_1 where $\dim(A_1) = n < \infty$. A graded right A-module $M = \bigoplus_{i\geq 0} M_i$ is called a point module (resp. line module) if:

(1) M is cyclic with $M = M_0 A$

& (2) $\dim_{\mathbb{k}}(M_i) = 1$ for all i (resp. $\dim_{\mathbb{k}}(M_i) = i + 1$ for all i).

Where's the geometry??

(1) $\Rightarrow A \twoheadrightarrow M$, so gdd $\Rightarrow A_1 \twoheadrightarrow M_1$. Kernel $K \subseteq A_1$,

(2) $\Rightarrow \dim_{\mathbb{k}}(K) = n - 1$ (resp. $n - 2$), $K^{\perp} \subseteq A_1^*$, $\dim_{\mathbb{k}}(K^{\perp}) = 1$ (resp. 2).

$\mathbb{P}(K^{\perp}) \subseteq \mathbb{P}(A_1^*)$, $\mathbb{P}(K^{\perp})$ = point in $\mathbb{P}(A_1^*)$ (resp. line in $\mathbb{P}(A_1^*)$).

Similarly, for other d-linear modules & truncated d-linear modules.

ATV 1989:

point modules are parametrized by a scheme, the point scheme.

Shelton & Vancliff 2000:

under certain hypotheses, line mods parametrized by a scheme, the line scheme. (For any $d \in \mathbb{N}$, d-linear modules are parametrized by a scheme.)

ATV used their geometry to classify certain types of algebras, where $n \leq 3$.

Definition ([Artin, Schelter] non-comm analogue of poly ring)

An assoc graded $\Bbbk$-algebra $A = \bigoplus_{i \geq 0} A_i$, with $A_0 = \Bbbk$, generated by A_1 is AS-regular of global dimension m if

- $\text{gldim}(A) = m < \infty$ &
- A has polynomial growth &
- (Gorenstein condition) a minimal projective resolution R of the trivial right module $\Bbbk_A$ consists of finitely generated mods & dualizing R yields a minimal projective resolution of the trivial left module ${}_A\Bbbk$.

Last condition is a symmetry property & replaces commutativity.

Examples

- Algebra S from above ($z_j z_i = \mu_{ij} z_i z_j$) is AS-regular, so, poly ring is AS-regular.
- If $\mathbb{k} = \mathbb{C}$, then many algebras from physics are AS-regular.
- For $\text{gldim}(A) = 1$, there is only $\mathbb{k}[x]$ = poly ring on 1 variable
- For $\text{gldim}(A) = 2$, there are only 2 types: generators x, y, 1 def relation f, where either
  $f = xy + \lambda yx$, $0 \neq \lambda \in \mathbb{k}$ (quantum affine plane), or
  $f = xy - yx - x^2$ (Jordan plane).
- For $\text{gldim}(A) = 3$, have 9 generic quadratic types & some with cubic relations. Generic types classified by ATV based on the point scheme X. $\exists\, \sigma \in \text{Aut}(X)$ such that the algebra = finite module over center iff $|\sigma| < \infty$; false, in general, for higher gldim.

For $\text{gldim}(A) = 4$, there are many examples, but no classification yet, not even for quadratic AS-regular algebras $\rightsquigarrow$ motivating open problem.

Global Dimension 4

Van den Bergh, mid-90s:
any quadratic algebra on 4 generators with 6 generic defining relations has 20 nonisom truncated point mods of length 3 (counted with multiplicity).
(Point scheme $\subseteq$ scheme of trunc pt mods of length 3.)

Van den Bergh, mid-90s:
any quadratic AS-regular algebra on 4 generators with 6 generic defining relations has a 1-parameter family of line mods.

Vancliff, Van Rompay & Willaert, mid-90s:
found quadratic AS-regular algebras on 4 generators with 6 defining relations with exactly 1 point mod (up to isom) & a 1-parameter family of line mods.

Shelton & Vancliff, 2000:
any quad alg on 4 gens with 6 def relns & a finite scheme of trunc point mods of length 3 $\Longrightarrow$ can recover the def relns from that scheme. (False, in general, for infinite scheme; no hypothesis of AS-regular nor of other homological data.)

Shelton & Vancliff, 2000:
any quad AS-reg algebra (+ a few other hypotheses) on 4 gens with 6 def relns & a 1-dimensional line scheme $\Longrightarrow$ can recover the def relns from the line scheme.

Last result suggests line scheme is important tool in gldim-4 case.

But are there any gldim-4 reg algs with exactly 20 nonisom point mods & a 1-diml line scheme?

Shelton & Tingey, 2001:
1st example of a quadratic AS-regular algebra on 4 generators with 6 defining relations with exactly 20 nonisom point mods & a 1-dimensional line scheme.

They used trial & error on a computer, not systematic. They & others could not reproduce the "trial & error" to find more such algebras.

In contrast, it is relatively easy to produce quadratic AS-regular algebras on 4 generators with 6 defining relations with exactly 20 nonisom point mods but with a 2-dimensional line scheme.

Can such algebras be generalized?

Graded Clifford Algebras

Definition ([Van den Bergh, Le Bruyn] char($\Bbbk$) $\neq 2$)

Let $M_1, \ldots, M_n \in M(n, \Bbbk)$ denote symmetric matrices. The *graded Clifford algebra* $C = C(M_1, \ldots, M_n)$, associated to $M_1, \ldots, M_n$, is defined to be the $\Bbbk$-algebra on degree-1 generators $x_1, \ldots, x_n$ and on degree-2 generators $y_1, \ldots, y_n$ with defining relations given by

(i) $x_i x_j + x_j x_i = \sum_{k=1}^{n} (M_k)_{ij}\, y_k$ for all $i, j = 1, \ldots, n$, and

(ii) y_k is central for all $k = 1, \ldots, n$.

Example ($n = 2$)

$$M_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \begin{array}{l} 2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2, \\ x_1x_2 + x_2x_1 = y_1 = x_1^2, \ \text{so} \end{array}$$

$$\frac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1x_2 + x_2x_1 - x_1^2\rangle} \twoheadrightarrow C. \qquad M_1 \leftrightarrow 2(t_1^2 + t_1t_2), \quad M_2 \leftrightarrow 2t_2^2.$$

Quadrics & Graded Clifford Algebras

GCAs are noetherian & are particularly nice due to the following result.

Theorem ([Aubry & Lemaire; Le Bruyn])

The GCA C is quadratic, regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ if and only if the quadric system in $\mathbb{P}^{n-1}$ associated to $M_1, \ldots, M_n$ is base-point free; in this case, C is a domain.

Example ($n = 2$)

$$M_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \leftrightarrow q_1 = 2(t_1^2 + t_1 t_2), \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \leftrightarrow q_2 = 2t_2^2.$$

$\mathcal{V}(q_1) \cap \mathcal{V}(q_2) = \emptyset \subset \mathbb{P}^1$. Thus, $C \cong \dfrac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1 x_2 + x_2 x_1 - x_1^2\rangle}$.

However, dim(line scheme of C) ≥ 2.

In order to generalize GCAs & the theorem, we needed to generalize the notions of

- symmetric matrix
- graded Clifford algebra
- quadratic form and quadric.

μ-symmetric Matrices

Definition ([Cassidy, Vancliff] $\Bbbk$ = arbitrary field)

Let $\mu = (\mu_{ij}) \in M(n, \Bbbk)$ be such that $\mu_{ij}\mu_{ji} = 1$ for all i, j such that $i \neq j$. A matrix $M \in M(n, \Bbbk)$ is called μ-symmetric if $M_{ij} = \mu_{ij} M_{ji}$ for all $i, j = 1, \ldots, n$.

Clearly,

$\mu_{ij} = 1$ for all $i, j \Rightarrow \mu$-symmetric = symmetric

$\mu_{ij} = -1$ for all $i, j \Rightarrow \mu$-symmetric = skew-symmetric (char($\Bbbk$) $\neq 2$).

Example

$n = 3$: $\begin{bmatrix} a & b & c \\ \mu_{21} b & d & e \\ \mu_{31} c & \mu_{32} e & f \end{bmatrix}$ is μ-symmetric.

Assumption

For the rest of the talk, assume $\mu_{ii} = 1 \;\; \forall\, i$ (& $\Bbbk$ still alg closed).

Graded Skew Clifford Algebras

Definition ($\text{char}(\Bbbk) \neq 2$ [Cassidy, Vancliff])

With μ as above, let $M_1, \ldots, M_n \in M(n, \Bbbk)$ denote μ-symmetric matrices. A *graded skew Clifford algebra*, associated to μ, $M_1, \ldots, M_n$, is a graded $\Bbbk$-algebra A on degree-1 generators $x_1, \ldots, x_n$ and on degree-2 generators $y_1, \ldots, y_n$ with defining relations given by:

(i) $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k$ for all $i, j = 1, \ldots, n$, and

(ii) the existence of a normalizing sequence $\{y_1', \ldots, y_n'\} \subset A_2$ that spans $\Bbbk y_1 + \cdots + \Bbbk y_n$.

Example

Skew polynomial rings on generators $x_1, \ldots, x_n$ with relations $x_i x_j = -\mu_{ij} x_j x_i$, for all $i \neq j$, are GSCAs.

Example ($n = 2$: quantum affine plane)

Let $M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2, \\ x_1x_2 + \mu_{12}x_2x_1 = 0,$$

so $$\frac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1x_2 + \mu_{12}x_2x_1\rangle} \twoheadrightarrow A.$$

Example ($n = 2$: "Jordan" algebra/plane)

Let $M_1 = \begin{bmatrix} 2 & 1 \\ \mu_{21} & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$,

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2, \\ x_1x_2 + \mu_{12}x_2x_1 = y_1 = x_1^2,$$

so $$\frac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1x_2 + \mu_{12}x_2x_1 - x_1^2\rangle} \twoheadrightarrow A.$$

Example

The quadratic AS-reg algebra found by Shelton & Tingey in 2001 that has gldim 4 & exactly 20 nonisom point mods and a 1-dimensional line scheme is a GSCA.

Remarks

- $x_j x_i + \mu_{ji} x_i x_j = \mu_{ji}(x_i x_j + \mu_{ij} x_j x_i) =$
  $$= \sum_{k=1}^{n} \mu_{ji} (M_k)_{ij}\, y_k = \sum_{k=1}^{n} (M_k)_{ji}\, y_k \quad \text{for all } i, j = 1, \ldots, n.$$
- Clearly, $\{$ GCAs $\} \subset \{$ GSCAs $\}$.
- GSCAs are noetherian.

It remains to generalize notions of quadatic form and quadric to try to relate properties of GSCA to some geometry.

To μ & $M_1, \ldots, M_n$, associate

1. the skew polynomial ring S on generators $z_1, \ldots, z_n$ with defining relations: $z_j z_i = \mu_{ij} z_i z_j$, for all $i \neq j$, and
2. the elements $q_k = z^T M_k z \in S_2$ where $z = [z_1 \ \ldots \ z_n]^T$.

Definition ([Cassidy, Vancliff])

We call any (nonzero) element of S_2 a quadratic form, and define the quadric, $\mathcal{V}(q)$, determined by any quadratic form q to be the set of points in $\mathbb{P}(S_1^*) \times \mathbb{P}(S_1^*)$ on which q and the defining relations of S vanish.

E.g., $(z_j z_i - \mu_{ij} z_i z_j)(\ (a_1, \ldots, a_n),\ (b_1, \ldots, b_n)\ ) = a_j b_i - \mu_{ij} a_i b_j \in \{0, 1\}$.

Definition ([Cassidy, Vancliff])

If $Q_1, \ldots, Q_m \in S_2$, we call their span a quadric system. A quadric system Q is said to be base-point free (BPF) if $\bigcap_{q \in Q} \mathcal{V}(q)$ is empty; Q is said to be normalizing if it is given by a normalizing sequence of S.

Theorem ([Cassidy, Vancliff])

A GSCA $A = A(\mu, M_1, \ldots, M_n)$ is a quadratic, regular algebra of global dimension n that satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ **if and only if** *the (non-commutative) quadric system associated to $M_1, \ldots, M_n$ is BPF & normalizing; in this case, A is a domain.*

This result allowed the production in [CV] of multi-parameter families of quadratic AS-regular algebras of gldim 4 with exactly 20 nonisom point mods and a 1-dimensional line scheme $\rightsquigarrow$ open problem: study line scheme of these algebras.

Example ($n = 2$: quantum affine plane)

$$M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \leftrightarrow q_1 = 2z_1^2, \qquad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \leftrightarrow q_2 = 2z_2^2.$$

$\{q_1, q_2\}$ = normalizing sequence in S.

$$\mathcal{V}(q_1) \cap \mathcal{V}(q_2) = \emptyset \;\Rightarrow\; A \cong \frac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1x_2 + \mu_{12}x_2x_1\rangle}.$$

Example ($n = 2$: "Jordan" algebra/plane)

$$M_1 = \begin{bmatrix} 2 & 1 \\ \mu_{21} & 0 \end{bmatrix} \leftrightarrow q_1 = 2(z_1^2 + z_1z_2), \qquad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \leftrightarrow q_2 = 2z_2^2,$$

$\{q_2, q_1\}$ = normalizing sequence in S.

$$\mathcal{V}(q_1) \cap \mathcal{V}(q_2) = \emptyset \;\Rightarrow\; A \cong \frac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1x_2 + \mu_{12}x_2x_1 - x_1^2\rangle}.$$

If $\mu_{12} = -1$, this is the usual Jordan algebra/plane.

Quadratic Quantum Planes

Previous slide $\implies$ all regular algebras of gldim 2 are GSCAs ($\text{char}(\mathbb{k}) \neq 2$).

Do GSCAs help in the classification of all reg algs? gldim 4? gldim 3?

Let D denote a quadratic AS-regular algebra of gldim 3.

The classification of such D depends on the point scheme X of D:

$X \subseteq \mathbb{P}^2$ & either X contains a line or it does not.

The latter case, splits into 3 subcases, so in total we have 4 cases:

- X contains a line
- X is a nodal cubic curve in $\mathbb{P}^2$
- X is a cuspidal cubic curve in $\mathbb{P}^2$
- X is an elliptic curve in $\mathbb{P}^2$.

Note: the classification of such D using GSCAs is work of myself with Manizheh Nafari & Jun Zhang. Our work attempts to classify <u>all</u> quadratic AS-regular algebras D of gldim 3 using GSCAs; not only the generic ones.

Theorem ([NVZ] char($\Bbbk$) $\neq 2$)

If X contains a line, then either D is a twist, by an automorphism, of a GSCA, or D is a twist, by a twisting system, of an Ore extension of a regular GSCA of gldim 2.

Theorem ([NVZ])

If X is a nodal cubic curve, then $D = \Bbbk[x_1, x_2, x_3]$ *with defining relations:*

$$\lambda x_1 x_2 = x_2 x_1, \qquad \lambda x_2 x_3 = x_3 x_2 - x_1^2, \qquad \lambda x_3 x_1 = x_1 x_3 - x_2^2,$$

where $\lambda \in \Bbbk$ and $\lambda^3 \notin \{0, 1\}$. Moreover, for any such λ, any quadratic algebra with these defining relations is regular & its point scheme X is a nodal cubic curve in $\mathbb{P}^2$.

Suppose char($\Bbbk$) $\neq 2$.

- *If $\lambda^3 \notin \{0, 1\}$, then D is an Ore extn of a regular GSCA of gldim 2;*
- *if $\lambda^3 = -1$, then D is a GSCA.*

Theorem ([NVZ])

X = cuspidal cubic curve in $\mathbb{P}^2$ if and only if

char$(\Bbbk) \neq 3$ and $D = \Bbbk[x_1, x_2, x_3]$ with def rels:

$$x_1x_2 = x_2x_1 + x_1^2, \quad x_3x_1 = x_1x_3 + x_1^2 + 3x_2^2, \quad x_3x_2 = x_2x_3 - 3x_2^2 - 2x_1x_3 - 2x_1x_2.$$

(Moreover, any such algebra is regular, even if char$(\Bbbk) = 3$.)

If char$(\Bbbk) \neq 2$ & X = cuspidal cubic curve, then D is an Ore extn of a regular GSCA of gldim 2.

It remains to consider X = elliptic curve in $\mathbb{P}^2$.

In [AS, ATV1], such algebras are classified into types A, B, E, H,

where some members of each type might not have an elliptic curve as their point scheme, but a generic member does.

Theorem ([NVZ] char($\Bbbk$) $\neq 2$)

Suppose X is an elliptic curve.

(i) *Regular algebras of type H are GSCAs.*

(ii) *Regular algebras of type B are GSCAs.*

(iii) *As in [AS, ATV1], regular algebras D of type A are given by*

$D = \Bbbk[x, y, z]$ with def rels:

$$axy + byx + cz^2 = 0, \quad ayz + bzy + cx^2 = 0, \quad azx + bxz + cy^2 = 0,$$

where $a, b, c \in \Bbbk$, $abc \neq 0$, $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$, char($\Bbbk$) $\neq 3$

& either $a^3 \neq b^3$, or $a^3 \neq c^3$, or $b^3 \neq c^3$.

- *If $a^3 = b^3 \neq c^3$, then D is a GSCA.*
- *If $a^3 \neq b^3 = c^3$ or if $a^3 = c^3 \neq b^3$, then D is a twist, by an automorphism, of a GSCA.*

In (iii), $a^3 \neq b^3 \neq c^3 \neq a^3$ is still open.

Open Problems

1. As stated on previous slide, it is still open whether or not type A with $a^3 \neq b^3 \neq c^3 \neq a^3$ is directly related to a GSCA.
2. Up to isomorphism & anti-isomorphism, type E consists of at most 1 algebra; it is still open whether or not this type is directly related to a GSCA.

If D is of type A or E, then its Koszul dual is the quotient of a regular GSCA; so, in this sense, such algebras are weakly related to GSCAs.

Open Problems (cont'd)

3. Can "cubic" regular algebras of gldim 3 be classified using GSCAs?
4. Can quadratic regular algebras of gldim 4 be classified using GSCAs? This is expected to need both the point scheme and the line scheme.

Open Problems (cont'd)

5. Can standard results on commutative quadratic forms and quadrics be extended to non-commutative quadratic forms and quadrics?
   E.g., Padmini Veerapen and I extended the notion of rank of a (commutative) quadratic form to non-commutative quadratic forms on n generators, where $n = 2,\ 3$; can this be done for $n \geq 4$?
   We used our notion of rank to extend results in [VVW] about point modules over GCAs to results about point modules over GSCAs.

6. Can standard results on symmetric matrices be extended or generalized to μ-symmetric matrices?

7. $\exists$ examples in [Stephenson, V] & in [CV] of AS-reg GSCAs of gldim 4 with N nonisom point modules, where $N \leq 20$ & $N \notin \{2,\ 19\}$; the excluded values do not arise for reg GCAs, but what about reg GSCAs?

8. [Stephenson, V] $\Rightarrow \exists$ quad AS-reg algs of gldim 4 with 2 nonisom point mods; what about 19?

Open Problems (cont'd)

9. As mentioned earlier, what is the line scheme of known quadratic regular algebras of gldim 4? Such as those in [CV]? Double Ore extensions? Generalized Laurent polynomial rings? etc

10. What is the line scheme of a generic quadratic AS-regular algebra of gldim 4?

Conclusion

There are many open problems in this rich subject, and some of them are very accessible to junior researchers.

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