MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: James Zhang

Talk Title: Open Questions in Noncommutative Algebra & Noncomm. Algebraic Geometry

Date: 01/24/13 **Time:** 9:15 am/ pm (circle one)

List 6-12 key words for the talk: GKdim, quantum symmetry, noncommutative space / algebra, AS regular algebras

Please summarize the lecture in 5 or fewer sentances: Introduce, discuss and propose open questions / problems about GK dimension, automorphisms of a noncommutative space / algebra, and constructions of such algebras, classification; AS regular algebras and their classification; homological identities of Nakayama automorphisms

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

NAGRT
MSRI
01/24/2013

Open Questions in NA (and NAG)

James Zhang

University of Washington
Seattle

Noncommutativity

Definition:

Two elements, or operations, f and g are commutative (resp. noncommutative) if

$$f \circ g = g \circ f \text{ (resp. } f \circ g \neq g \circ f).$$

Example: of noncommutativity

$$(\text{walk 4 ms}) \circ (\text{turn } 90^\circ \text{ degs}) \neq (\text{turn } 90^\circ \text{ degs}) \circ (\text{walk 4 ms})$$

Noncommutativity	$\Longrightarrow$	natural
	$\Longrightarrow$	interesting
	$\Longrightarrow$	mysterious
	$\Longrightarrow$	hard to imagine
	$\Longrightarrow$	challenging

Our goal and general setup

Our Goal is to talk about **open questions** in noncommutative algebra that are closely related to noncommutative algebraic geometry.

General setup:

k be a base field, and everything is over k.

Star system for today's questions:

* could be easy

** not easy

*** difficult, but we want to give a try

**** too difficult

***** not a good idea to work on it

Dimension

Q1.1: (3*) What is the meaning/definition of the "dimension" of a noncommutative algebra or a noncommutative space?

Q1.2: (1*) Are there many different "dimension"s for an object in the noncomm world?

One possibility is Gelfand-Kirillov dimension.

To simplify my presentation, I will consider connected graded locally finite algebras.

$A = k \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ with $\dim_k A_{<\infty}$.

$$\operatorname{GKdim} A := \lim_{n\to\infty} \log_n\Big(\sum_{i=0}^{n} \dim_k A_i\Big).$$

The Hilbert series of A is defined to be

$$H_A(t) = \sum_{i=0}^{\infty} (\dim_k A_i) t^i.$$

GKdim

Lemma: If $H_A(t)$ is a rational function of the form $\frac{f(t)}{(1-t)^d g(t)}$, for some d, where $f(1)g(1) \neq 1$, and if the roots of $g(t)$ are root of unity, then $\operatorname{GKdim} A = d$.

Example: Let $A(1) = k[x_1, \cdots, x_n]$ (or let q be a scalar in k and $A(q) = k_q[x_1, \cdots, x_n]$ with $x_j x_i = q x_i x_j$ for all $i < j$).

$\Rightarrow H_A(t) = \frac{1}{(1-t)^n}$ where $A = A(1)$ or $A(q)$.

$\Rightarrow$ By Lemma, $\operatorname{GKdim} A = n$.

$\Rightarrow$ In the commutative case (when $q = 1$), n is the (Krull) dimension of $A(1)$.

$\Rightarrow$ So n should also be the dimension of the noncommutative algebra $A(q)$.

Examples of NA

1. Let $q \in k^{\times} := k \setminus \{0\}$. The q-skew polynomial ring $A(q) := k_q[x_1, \cdots, x_n]$ is generated by $x_1, \cdots, x_n$ subject to the relations

$$x_j x_i = q x_i x_j$$

for all $i < j$. If $q \neq 1$, $x_j x_i \neq x_i x_j$ of $i \neq j$. We have seen that $\operatorname{GKdim} A(q) = n$ and $A(q)$ is generated by n elements.

2. Down-up algebra (a special case):
Let D be the algebra generated by $d :=\downarrow$ and $u :=\uparrow$ and subject to the relations

$$d^2 u = u d^2; \quad u^2 d = d u^2.$$

This is a special down-up algebra in the work of [Benkart, Roby] and an AS regular algebra of type (S1) in the work of [Artin, Schelter].

Note that $du \neq ud$. By an (easy) computation, $H_D(t) = \frac{1}{(1-t)^3(1+t)}$. So $\operatorname{GKdim} D = 3$. But, D is generated by 2 elements.

Questions about GKdim

<u>Q1.3:</u> (3*) If $\mathsf{GKdim}\, A < \infty$, and assume that A is noetherian, is then $\mathsf{GKdim}\, A$ an integer?

[Bergman, Smoktunowicz] True if $\mathsf{GKdim}\, A < 3$.

Noetherian: left/right ideals of A satisfy acc.

<u>Q1.4:</u> (3*) If A is noetherian, is $\mathsf{GKdim}\, A$ finite?

<u>Q1.5:</u> (2*) If A is noetherian, is $H_A(t)$ a rational function?

(Above true for commutative or PI algebras.)

<u>Q1.6:</u> (1*) How to define GKdimension for a noncommutative space?

Symmetry

Q2.1: (2*) What is an "automorphism" of a noncomm algebra or a noncomm space?

Automorphisms of a commutative algebra or a variety should be the usual automorphisms. But automorphisms of a noncommutative algebra or a noncommutative variety (or a quantum space) could be quantum automorphisms. What are the "quantum automorphisms"?

Example:

Only Hopf algebra actions on $k[x_1, x_2]$ are group actions (acting as usual automorphisms) [Chan-Walton, Etingof-Walton].
There are many "quantum automorphisms" (e.g., non-group Hopf algebra actions) on $k_{-1}[x_1, x_2]$ [Chan-Kirkman-Walton, others]

Questions about Aut(A)

Let Aut(A) denote the group of algebra automorphisms of A.

Q2.2: (5*) Describe Aut($k[x_1, x_2, x_3]$) = Aut($\mathbb{A}^3$).

$\text{Aut}(k[x_1]) = \{\text{linear auts}: \quad x_1 \to ax_1 + b\}$

[Jung, Van der Kulk] Aut($k[x_1, x_2]$) is tame.

(namely, it is generated by linear auts and triangular auts ($x_1 \to x_1, x_2 \to x_2 + f(x_1)$).)

Lem: [Alev, Dumas, Jordan, Goodearl, others] If q is not a root of 1, then $\text{Aut}(A(q)) = (k^\times)^n$.

Q2.3: (1*) What is Aut($k_q[x_1, \cdots, x_n]$) when q is a root of unity and $q \neq \pm 1$ (for all n)? (Known for small n)

Quantum symmetry

Q2.4: (1*) What is $\mathrm{Aut}(k_{-1}[x_1, \cdots, x_n])$ when $n \geq 3$ is odd? (Known for even n)

Noncommutative objects are more rigid, therefore less symmetric (group-symm). But noncommutative objects have quantum symmetry.

Lemma: [Chan-Walton] If q is not a root of 1, the only (degree preserving) Hopf algebra actions on $k_q[x_1, \cdots, x_n]$ is the group action.

Q2.5: (2*) Assume q is a root of 1. Classify all Hopf algebra actions on $k_q[x_1, \cdots, x_n]$ (for all $n \geq 3$).

Q2.6: (3*) Is there any new "symmetry" beyond quantum symmetry (i.e. Hopf algebra actions)?

Remark: We may replace $k_q[x_1, \cdots, x_n]$ by any AS regular algebra of dimension at least 3.

Ramras' conjecture

An algebra A is called **local** if the Jacobson radical of A is a maximal left ideal of A.

We say A is **regular** or has finite global dimension $< d$, if projdim $M < d$ for all modules.

Theorem: [Auslander, Buchsbaum, Nagata, etc] If A is noetherian commutative regular local, then A is a (UF) domain.

Ramras showed that noetherian local with global dimension ≤ 3 is a domain, and he conjectured

C3.1: (Ramras' conjecture 4*) Every noetherian noncommutative local regular algebra is a domain.

Theorem: [Brown+others, Stafford] C3.1 holds if A is PI.

Algebra $\rightleftharpoons$ Geometry

Examples:

Commutative Algebra $\rightleftharpoons$ Algebraic Geometry

category of affine comm. algs is dual to category of affine schemes

Operator Algebra $\rightleftharpoons$ Differential Geometry

category of comm. C^*-algs is dual to category of cp. top. spaces

Noncomm. Algebra $\rightleftharpoons$ Noncomm. Alg. Geo.

category of noncomm. algs is dual to category of noncomm. spaces

Conclusion:

Algebra and Geometry support each other.

Noncomm algebras serve as affine/local models for noncomm spaces.

So you might translate Q in noncomm alg. to Q in noncomm geo.

Construction of algs

Free algebras: **commutative free algebras** are polynomial rings, denoted by $k[V]$ where V is a vector space; **noncommutative free algebras** are denoted by $k\langle V\rangle$.

Every commutative algebra is a **factor** of a polynomial ring and every noncommutative algebra is a **factor** of a noncommutative free algebra.

One could take **subalgebra** of a given algebra.

More constructions:

Matrix algebra over A is $B := M_n(A)$. If $n > 1$, B is noncommutative even if A is.

Path algebra of a quiver.

Smash product (or tensor product) $A\#H$ if a Hopf algebra H acts on A.

More constructions

Consider connected graded rings.

IOE: Iterated Ore extension is defined by

$A^0 = k, A^1 = k[x_1]$, and

$A^n := A^{n-1}[x_n; \sigma_n, \delta_n]$ where $\deg X_n = 1$, σ_n is a graded algebra automorphism of A^{n-1}, and δ_n is a σ_n-derivation of A^{n-1}.

HUL: Homogenization of universal enveloping algebra over a finite dimensional Lie algebra L is denoted by $H(L)$, which is generated by L and a central variable t subject to the relations

$$ab - ba = [a, b]t$$

where $a, b \in L$ and $[-,-]$ is the Lie product in L.

THC: [Artin, Van den Bergh] Twisted homogeneous coordinate ring is denoted by $B(X, \sigma, \mathcal{L})$ where X is a projective scheme, σ is an automorphism of X, and $\mathcal{L}$ is a σ-ample line bundle over X.

IOE, HUL, THC are noetherian connected graded algebra with finite GKdim.

Qs about constructions

Q4.1: (2*) Find new and natural constructions for noncommutative algebras/spaces.

Q4.2: (5*) How can we "construct" all noetherian connected graded algebras of finite GKdimension?

Q4.2': (5*) Can we classify all noetherian connected graded algebras of finite GKdimension?

Q4.3: (1*) Looking for invariants that help us to understand the structure of connected graded algebras.

AS regular algebras

<u>Definition:</u> A connected graded algebra A is called Artin-Schelter regular (or AS regular) if
(i) A is regular (i.e. gl.dim$A = d < \infty$),
(ii) $\mathsf{GKdim}\, A < \infty$,
(iii) $\text{Ext}^i_A(k, A) = \begin{cases} 0 & i \neq d \\ k(l) & i = d \end{cases}$.

AS regular algebras play an important role in noncommutative projective geometry. If A is noetherian and AS regular, Proj A has most homological properties that QCoh($\mathbb{P}^n$) has. (For example, Proj A has Serre functor, etc)

<u>Theorem:</u> [Rogalski, Sierra] Not every noetherian regular algebra of f. GKdim is AS regular.

IOE and HUL are AS regular.

<u>Theorem:</u> [Artin, Tate, Van den Bergh] Ramras conjecture holds for AS regular algebras of global dimension 4. (Still open for 5 or higher).

AS regular algebras of dimension ≤ 3

dim $= 0$: k. dim $= 1$: $k[x]$.

dim $= 2$: Suppose $k = \overline{k}$, then $k_q[x_1, x_2]$ and $k_J[x_1, x_2] := k\langle x_1, x_2\rangle/(x_2x_1 = x_1x_2 + x_1^2)$. Both of these are IOE.

dim $= 3$: Artin, Schelter, Tate, Van den Bergh classified all AS regular algebras of dimension (=global dimension) 3. (14 families)

Not all AS regular algebra of dimension 3 is IOE. (Sklyanin algebra of dimenison 3 is not IOE).

If A is AS regular of dimension 3, then Proj A is a quantum plane.

(The dimension of a projective space is 1 less than the dimension of corresponding affine space.)

Qs about AS reg algs

P5.1: (4*) Classify all AS regular algebras of global dimension 4.

Many people have made contributions: Stafford, Smith, Van den Bergh, Tate, Vancliff, Shelton, Wu, Palmieri, Lu, Rogalski, more...

Q5.2: (2*) Is every AS regular algebra noetherian?

ASTV proved this is true for dimension 3.

Q5.3: (2*) If A is a noetherian AS regular over a finite field, is then A finitely generated over its center? (True for IOE and HUL)

Q5.4: (3*) [Kirkman, Kuzmanovich] Let A be a noetherian AS regular algebra. Under what conditions, is the fixed subring A^G AS regular? (We are asking for a noncommutative version of Shephard-Todd-Chevalley Theorem.)

Nakayama automorphism I: Classical case

[Nakayam] A finite dimensional algebra A is *Frobenius*, if $\operatorname{injdim} A = 0$, or there there is a nondegenerate associative bilinear form

$$\langle -, - \rangle : A \times A \to k.$$

This is equivalent to the existence of an isomorphism A-bimodule $A^* \cong {}^{\mu}A^{1}$ where μ is an algebra automorphism of A. The automorphism μ can also be described by equation

$$\langle a, b \rangle = \langle \mu(b), a \rangle$$

for all $a, b \in A$.

Example: Let $A = k\langle x, y \rangle / (x^2, y^2, yx - qxy)$. Then $A = k \oplus kx \oplus ky \oplus kxy$. The bilinear form is defined by $\langle a, b \rangle = pr_{(kxy)}(ab)$. Using this bilinear form, it is easy to check that

$$\mu : x \mapsto qx, y \mapsto q^{-1}y.$$

Nakayama automorphism II

Definition: An algebra A is called skew Calabi-Yau (or skew CY, for short) if (i) A is homologically smooth, that is, A has a projective resolution in the category A^e-Mod that has finite length and each term in the projective resolution is finitely generated, and (ii) there is an integer d and an algebra automorphism μ of A such that

$$\mathrm{Ext}^i_{A^e}(A, A^e) = \begin{cases} 0 & i \neq d \\ {}^1A^{\mu} & i = d \end{cases}, \qquad \text{(E1.1)}$$

as A-bimodules, where 1 denotes Id_A. ♣

If (E1.1) holds (even without (i)), then μ is called the **Nakayama automorphism** of A. Note that μ (if exists) is unique up to inner automorphisms of A. Also write μ_A for μ.

A is called CY if $\mu_A = Id$.

If A is finite dimensional, then this agrees with the Nakamaya automorphism given in the previous page.

Existence

Let H denote a Hopf algebra. Then we have maps $(m, u, \Delta, \epsilon, S)$ associated to H. Here S is the antipode of H.

Theorem: [Brown] Let H be a noetherian Hopf algebra satisfying AS condition. Then

$$\mu_H = S^2 \circ \Xi^l_{\int^l}. \qquad \text{(HI1)}$$

where $\int^l$ is the left integral of H.

Theorem: [Yekutieli, Rogalski, Reyes, others] If A is AS regular, μ_A exists uniquely.

Examples:

$\mu_D : d \to -d, u \to -u.$

$\mu_{A(q)} : x_i \to q^{2i-n-1} x_i$ for $i = 1, 2, \cdots, n$.

Homological identities

Theorem: [Chan-Walton] Let A be a Koszul AS regular algebra. Let H be a Hopf algebra coacting on A inner-faithfully such that $hcodet \in H$ is the homologial codeterminant of the H-coaction. Then

$$\eta_{\mu_A^\tau} = S^2 \circ \eta_{hcodet} \qquad \text{(HI2)}$$

where η_{hcodet} is the automorphism of H defined by conjugating $hcodet$ and $\eta_{\mu_A^\tau}$ is the automorphism of H given by conjugating by the transpose of the corresponding matrix of μ_A.

Q6.1: (2*) Can we unify Brown's identity (HI1) with Chan-Walton's identity (HI2)?

Q6.2: (2*) If A is noetherian and AS Gorenstein, is hdet $\mu = 1$? $\leftarrow$ Another HI.

Theorem: [Rogalski, Reyes] Yes to Q7.2, if A is AS regular and Koszul.

Artin's conjecture

P7.1: Artin's conjecture on division algebras of transcendence degree 2 and noncommutative projective surfaces (come to Sierra's talk).

Thank you very much