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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Sarah Witherspoon

Talk Title: Poincaré - Birkhoff - Witt Theorems

Date: 01 / 24 / 13 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: PBW Theorems, PBW bases, quantum groups, finite dimensional Hopf algebras, Drinfeld Hecke algebras, Koszul algebras

Please summarize the lecture in 5 or fewer sentances: discuss the classical PBW Theorem, how it contributes to the structure of a Lie algebra. Introduce results on algebras that have PBW bases and how that helps to prove finite generation of the cohomology. Propose open problems in this direction with the conjecture of Etingof and Ostrik '04. Discuss Drinfeld Hecke algebras and open questions for generalization.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

01/24/13 **Sarah Witherspoon**: "Poincaré – Birkhoff – Witt (PBW) Theorems"
11:00 am

1. Classical PBW Theorem: (Poincaré 1900, Birkhoff 1937, Witt 1937)

Defn: A Lie algebra $\mathfrak{g}$ is a vector space over a field k with a bilinear map
$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying properties ....
$(x, y) \mapsto [x, y]$

Example: $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, 2×2 matrices of trace 0
basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$[x, y] = xy - yx$

Defn: Its universal enveloping algebra
$$U(\mathfrak{g}) = T(\mathfrak{g}) / (x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g})$$
$k \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots$
with multiplication is "$\otimes$"
$T(\mathfrak{g})$ = tensor algebra

Example: $U(\mathfrak{sl}_2(\mathbb{C})) = \mathbb{C}\langle e, f, h \mid ef - fe = h,\ he - eh = 2e,\ hf - fh = -2f \rangle$

Theorem: (PBW) Let $\{x_i \mid i \in I\}$ be a totally ordered basis of $\mathfrak{g}$. The set $\{x_{i_1} x_{i_2} \cdots x_{i_n} \mid i_1 \leq i_2 \leq \cdots \leq i_n\}$ is a basis of $U(\mathfrak{g})$.
Moreover, $\mathrm{gr}\, U(\mathfrak{g}) \cong S(\mathfrak{g})$
symmetric algebra
(i.e. polynomial ring in a basis of $\mathfrak{g}$)

Example: $U(\mathfrak{sl}_2(\mathbb{C}))$ with basis $\{e^a h^b f^c \mid a, b, c \geq 0\}$

② <u>Quantum groups and related algebras</u>:

<u>Example</u>: let $q \in \mathbb{C}^\times$

$$U_q(sl_2) = \mathbb{C}\left\langle e, f, k^{\pm 1} \;\middle|\; ef - fe = \frac{k-k^{-1}}{q-q^{-1}},\ ke = q^2 ek,\ kf = q^{-2} fk \right\rangle$$

If q is a primitive n^{th} root of 1:

$$u_q(sl_2) = U_q(sl_2) / (e^n, f^n, k^n - 1)$$ finite dimensional

<u>Theorem</u>: $U_q(\mathfrak{g})$ has a PBW basis

e.g. $U_q(sl_2)$ basis: $\{ e^a k^b f^c \mid a, c \geq 0,\ b \in \mathbb{Z} \}$

$gr\, U_q(sl_2)$ is a skew polynomial ring

<u>Theorem</u>: (Ginzburg-Kumar 1993) $H^*(u_q(\mathfrak{g}))$ is finitely generated, where $H^*(u_q(\mathfrak{g})) = \mathrm{Ext}^*_{u_q(\mathfrak{g})}(\mathbb{C}, \mathbb{C})$

<u>Note</u>. $u_q(\mathfrak{g})$ is a Hopf algebra

$H^*(u_q(\mathfrak{g}))$ is graded commutative

↳ Proof of Thm. uses PBW basis.

More generally:

<u>Theorem</u>: Let A be a finite dimensional (f.d.) pointed Hopf algebra with abelian group of grouplike elements. Then A has a (restricted) PBW basis.

(Andruskiewitsch-Schneider 2010 classification of such Hopf algs.)

↳ such Hopf algs. look like quantum groups.

<u>Theorem</u>: (Mastnak - Pevtsova - Schauenburg - Witherspoon 2010)

A as above. Then $H^*(A)$ is finitely generated.

As a consequence, one may apply "varieties for modules".

- Open Problems:
  + Prove or find a counterexample to
    conjecture (Etingof - Ostrik 2004): If A is a finite dimensional Hopf algebra, then $H^*(A)$ is finitely generated.
  + One may use further PBW bases for other classes of examples to prove this
  + Prove PBW Theorems for more general classes of examples.

(3) Drinfeld Hecke algebras and related algebras:

Let G be a finite group, (k be a field)

V be a f.d. kG-module

$\rightsquigarrow$ G acts on $T(V)$ and on $S(V)$ by automorphisms

$T(V)$: tensor alg, $S(V)$: symmetric alg.

Note: $S(V) \cong T(V)/(v\otimes w - w\otimes v \mid v, w \in V)$

$T(V) \rtimes G := \overset{v.s.}{\cong} T(V) \otimes_k kG$, as vector space

$\hookrightarrow$ has multiplication $(r\otimes g)(s\otimes h) = r(g(s)) \otimes gh$

$\forall\, r, s \in T(V)$ and $g, h \in G$

Let $K: V\otimes V \to kG$

$$H_K := T(V)\rtimes G \big/ (v\otimes w - w\otimes v - K(v\otimes w) \mid v, w \in V)$$

Theorem: Explicit conditions on K $\Longleftrightarrow$ $\operatorname{gr} H_K \cong S(V)\rtimes G$

(i.e. H_K has a "PBW basis")

These algebras were introduced by various people in different contexts

Drinfeld 1986, Lusztig 1988 (termed "graded Hecke algebras")

Ram - Shepler 2003

Cherednik 1995, Etingof - Ginzburg 2002 (termed "symplectic reflection algebras")

Griffeth 2010 (char. p)

More generally (Shepler - Witherspoon)

G acts on a Koszul algebra, k is of arbitrary characteristic, κ can take values in $kG \oplus (kG \otimes V)$

↳ techniques from Braverman - Gaitsgory)

- Open Questions:
  + Representations in more general contexts
  + Generalize to N-Koszul algebras
  + Generalize to Hopf algebra actions on $S(V)$, $S_q(V)$, other Koszul algebras.

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