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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Catharina Stroppel

Talk Title: Kazdhan - Lusztig polynomials, geometry and categorification

Date: 01 / 24 / 13 **Time:** 1 : 30 am / pm (circle one)

List 6-12 key words for the talk: Kazdhan - Lusztig polynomials, KL basis, Hecke algebras, Verma modules

Please summarize the lecture in 5 or fewer sentances: define Kazdhan - Lusztig polynomials and discuss their appearance in representation and geometry. Indicate their role in commutative geometry (Soergel) and noncommutative (KL, Soergel), and in the concept of categorification.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

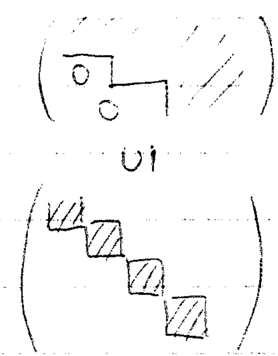
☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

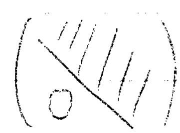
01/24/13 Catharina Stroppel: "Kazdhan – Lusztig polynomials,
1:30 pm geometry and categorification."

– Classical problem: $\mathfrak{g}$ semisimple, reducible complex Lie algebra
$\cup$
$\mathfrak{p}$ parabolic
$\cup$
$\mathfrak{l}$ levi

example: $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$
$\cup\!\vert$



V is irreducible $\mathfrak{l}$-module
$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V = \Delta(V)$$
generalized Verma module

Structure?
Special case: $\mathfrak{p} = \mathfrak{l} =$ 

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{l})} \mathbb{C}_\lambda = \text{ordinary Verma module}$$

+ Usual approach: $\rightsquigarrow$ geometry ($\mathcal{D}$-modules / perv. sheaves)
+ Problem: No classification of V's! Annihilators are
And composition factors might have infinite multiplicative
or infinitely many ($\rightarrow$ examples of Stafford)

Do new (categorification) ~~require~~ techniques help?
Now: (W, S) coxeter group.
In this talk, $W = S_n$
$$\mathbb{Z}[W] \xrightarrow[\text{deform}]{} \mathcal{H} = \mathcal{H}(W, S) \text{ Hecke algebra over } \mathbb{Z}[q, q^{-1}]$$

generators H_{s_i}, $s_i \in S$

relations: $H_{s_i} H_{s_j} = H_{s_j} H_{s_i}$ $\quad |i-j|>1$

$$H_{s_i} H_{s_{i+1}} H_{s_i} = H_{s_{i+1}} H_{s_i} H_{s_{i+1}}$$

$$H_{s_i}^2 = 1 + (q^{-1} - q) H_{s_i} \quad (\text{in particular } H_x \text{ def.}, \forall x \in W)$$

Have nontrivial involution $\varphi: H_{s_i} \longmapsto H_{s_i}^{-1}$

$$q \longmapsto q^{-1}$$

<u>Theorem</u> (Kazdhan-Lusztig '79)

$\forall x \in W$, $\exists!$ $\underline{H}_x$ s.t. $\varphi(\underline{H}_x) = \underline{H}_x$

$$\underline{H}_x \in H_x + \sum_{y<x} q\mathbb{Z}[q] H_y$$

KL-basis elements $\qquad$ KL-polynomials $P_{x,y}(q)$

<u>ex</u>: $\underline{H}_e = 1$

$\underline{H}_{s_i} = H_{s_i} + q$

<u>Facts</u>: . no closed formula (except of small cases)

. extends to KL-basis of induced modules

$$N^P = \mathcal{H}(W) \otimes_{\mathcal{H}(W_P)} \mathrm{sgn}^{|w|}$$

Explicit formulas for $W_P = S_i \times S_{n-i}$, (sgn)

Standard basis $\longleftrightarrow (W/W_P)^{\text{short}} \longleftrightarrow \left\{ \begin{array}{l} \text{sequences with } i \wedge\text{'s} \\ n-i \vee\text{'s} \end{array} \right\}$

$$e \longleftrightarrow \underbrace{\wedge\wedge\wedge\wedge}_{i}\underbrace{\vee\vee\vee\vee}_{n-i}$$

$$w \longleftrightarrow w(\wedge\wedge\wedge\wedge\vee\vee\vee\vee)$$

KL-basis $\longleftrightarrow$ cup diagram associated with the sequence with max. # of anticlockwise cups

$$\underline{N}_x \longleftrightarrow \underline{N}_x$$

e.g. $N_y \wedge\wedge\vee\wedge\vee\wedge\vee\wedge\vee$ $\qquad P_{x,y}(q) = \begin{cases} q^c, & c \text{ \# clockwise cups} \\ 0, & \text{if } N_y, \underline{N}_x \text{ not oriented} \end{cases}$

N_x | |

$\rightsquigarrow$ basic fact why Khovanov homology

- Interpretation in terms of commutative geometry (Soergel)

$W \hookrightarrow V = \mathbb{C}^n$

$S = S(V^*)$ $\qquad x \in W \rightsquigarrow$ coherent sheaf on $V \times V$

$= $ regular functions on $\{(xv, v) \mid v \in V\} = S^x$

$S_i \rightsquigarrow B_{s_i} := S \otimes_{S^{s_i}} S$ $\quad$ (filter by S, S^{s_i})

Theorem: (Soergel)

$$K_0 \left(\begin{array}{l} \text{add. (tensor) category generated by} \\ \text{all } B_{s_i}\text{'s and closed under sums} \\ \text{and summands} \end{array} \right) = \mathcal{H}$$

indecomposable bimod. $B_x \longleftrightarrow \underline{H}_x$

$S^x \longleftrightarrow H_x$

Q: Is there a commutative version for N^P? Connections to DT? Fukaya?

- Noncommutative: $A = \mathrm{End}_S\left(\bigoplus_{x \in W} B_x \otimes_S S/m \right)$

Known A-modules: $\rightarrow \mathcal{O}_0(\mathfrak{gl}_n)$, BGG

$\rightarrow A = \mathrm{End}^\bullet\left(\bigoplus \overset{\text{hypercohom.}}{\mathbb{H}}(\text{simple perv. sheaves on } GL(n,\mathbb{C})/B)\right)$

<u>Theorem</u>: (KL, Soergel)

$K_0(A\text{-gmod.}) \xrightarrow{\simeq} \mathcal{H}$ as a $\mathcal{H}$-module

indec. proj object $\longleftrightarrow \underline{H}_x$

Verma modules $\longleftrightarrow {}_{s_i}H_x$

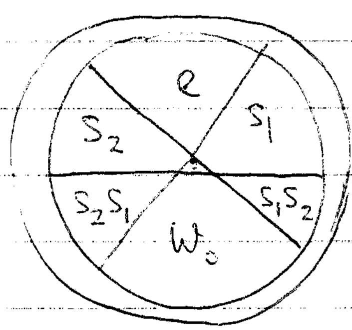
simple objects $\longleftrightarrow$ dual KL basis $\underline{H}^x_x$

<u>Question</u>: Can we understand arbitrary $\mathcal{H}$-modules?

- How to describe irreducible modules?
  (KL defined integral version (= cell modules) of irred. modules)
  Say $y \leq x$ if B_y occurs as a summand in FB_x
  where F = product of B_{s_i}'s

$$x \sim y \iff x \leq y,\ y \leq x$$

S_3



Given $x \rightsquigarrow \mathcal{H}_{\leq x} := \mathrm{Span}\{\underline{H}_y \mid y \leq x\}$

$\mathcal{H}_{\leq x} / \mathcal{H}_{< x}$ is simple $\mathcal{H}$-module

Categorified

<u>Theorem</u>: (Mazorchuk, S.)

Take $x \in W$. Take Serre subcategory generated by $L(y)$, $y \leq x$ and quotient by Serre subcat. gen. by $L(y)$, $y < x$

<u>Remark</u>: Resulting category doesn't depend on choice of x.
↳ Using 2-category theory.
Again, simple modules have KL-bases / dual KL-bases corresponding to proj. / simple

- Explicit description of these categories?
Known. (Khovanov - L., and, BK, BS) equivalent to blocks of cycl. Hecke algebra
+ <u>Special case</u>: Simple modules labelled by partitions with 2-rows
<u>Observation</u>: GK dim $\longleftrightarrow$ Lusztig's a function
$\longleftrightarrow$ dimension of a Springer fibre

<u>Example</u>: $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}$ irred. representation appears as a submod. in N^{ρ}, where $W^{\rho} = S_2 \times S_2 \subset S_4$
Consider the Springer fibre attached to nilpotent $\begin{array}{c} \left(\begin{array}{cc|cc} & 1 & & \\ & 0 & & \\ \hline & & 0 & 1 \\ & & & 0 \end{array}\right) \begin{array}{c} 2 \\ \\ 2 \end{array} \end{array}$

$$\left\{(\overset{\mathcal{N}}{\overset{\Downarrow}{x}}, \overset{\swarrow}{e}) \mid x \in \mathcal{C}\right\} \quad \{F_1 \subseteq \ldots \subseteq F_n\} \; xF_i \subseteq F_{i-1}\}$$

$$\begin{array}{ccc} \pi \big\downarrow & & \overset{\shortparallel}{\pi^{-1}(x)} \end{array}$$

$\mathcal{N} = \{$nilpotent elements in $\mathfrak{g}\} \ni x$
Springer fibre has Components $\pi^{-1}(x) = \mathbb{P}^1 \times \mathbb{P}^1 \cap$ Hine. surface
$$F_1 \subseteq \underset{\underset{x^{-1}(F_1)}{\shortparallel}}{F_2} \subseteq F_3 \subseteq \mathbb{C}^x$$

Categorification of irred. modules $=$ H-mod, where
$$H = \bigoplus_{\substack{(C_i, C_j) \\ \text{components in Springer fibre}}} H^*(C_i \cap C_j)$$ with a convolution product / Floer homology product

⇝ "Symplectic Khovanov homology".

- Solution of original problem:
  Interpret gen. Vermas as standard objects in a category with $K_0 = \mathcal{H}(W) \otimes_{\mathcal{H}(W_p)}$ irred. modules

(joint with Mazorchuk.)
(rough structure of generalized Verma modules)
