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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Graham Leuschke

Talk Title: What Should a Noncommutative Resolution of Singularities Be?

Date: 01/24/13 **Time:** 3:15 am / pm (circle one)

List 6-12 key words for the talk: non-commutative resolution, singularities, R-order, non-singular, Gorenstein, crepant resolution, symmetric, gl.dim

Please summarize the lecture in 5 or fewer sentances: Analyze the problem with the usual resolution of singularities of a variety X, then modify and restrict the ring R to define a non-commutative resolution of singularities of a Gorenstein local ring R. Equivalences of non-comm. crepant resolution of singularities of R and result about rational singularities. Introduce Bondal-Orlov's conjecture and open problems on crepant resolutions.

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(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
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☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

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☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

01/24/13 Graham Leuschke : "What Should a Noncommutative
3:15pm Resolution of Singularities Be ?"

Defn: A (usual) resolution of singularities of a variety X is $\pi : Y \to X$ such that
i) π is birational (isomorphism on function fields)
ii) π is proper (e.g. projective)
iii) ~~π~~ Y is non-singular.

– Petty complaints about the above definition :
1) Resolutions of singularities of affine varieties are usually not affine. Instead, if $X = \operatorname{Spec} R$, we end up looking at Rees rings $R[It]$. In fact, $\operatorname{Proj} R[It]$

2) It's known to exist in char. 0 [Hironaka] It's extremely complicated

3) Given the trend in "modern" algebraic geometry to focus on categories $\operatorname{coh} X$, $\operatorname{Qcoh} X$, $D^b(\operatorname{coh} X)$, we'd like to control these categories under π.

Maybe we could stick to affine varieties ?
↳ No. $\mathbb{C}[x,y,z]/(x^3+y^2+z^2)$ has no module - finite regular birational algebras

– If we focus on derived categories, then we know that [Rickard] : $D^b(-)$ is invariant under Morita equivalence.

e.g. $D^b(\mathrm{mod}\, R) \simeq D^b(\mathrm{mod}\, \mathrm{Flat}_n(R))$

$\Rightarrow$ $D^b(-)$ is <u>blind</u> to commutativity.

Let's try to rescue the naive hope above by allowing module-finite algebras $R \to \Lambda$ with Λ non-commutative ($R \hookrightarrow Z(\Lambda)$)

- <u>Mimic</u> the definition at the beginning:

i) Birational: Could ask $\Lambda \otimes_R K \cong K$, where $K = Q(R)$. quotient field

But we should ask for Morita equivalence, so

$$\Lambda \otimes_R K = \mathrm{Mat}_n(K)$$

ii) Assume module-finite

iii) Non-singularity: $\mathrm{gl.dim}\, \Lambda < \infty$

But finite global dimension is not well behaved for non-comm. rings

$\hookrightarrow$ To avoid this bad behavior, <u>restrict</u> R.

+ Assume R is a Cohen-Macaulay (CM) local ring (in a moment, R will be Gorenstein)

<u>examples</u>: rings of invariants, determinants of hypersurfaces

+ Assume Λ is an <u>R-order</u>: Λ is maximal CM as an R-module (free over polynomial subring)

+ Assum $\mathrm{gl.dim}\, \Lambda = \dim R$ (Krull dim.)

(Λ is a <u>non-singular</u> R-algebra)

$\Rightarrow$ These assumptions lead to satisfactory behavior:

<u>Prop:</u> For R Gorenstein local and Λ an R-order, TFAE:

1) $\mathrm{gl.dim}\, \Lambda = \dim R$

2) $\mathrm{gl.dim}\, \Lambda \leq \dim R$

3) $\mathrm{gl.dim}\, \Lambda < \infty$ and $\omega_\Lambda \overset{\text{defn}}{=} \mathrm{Hom}_R(\Lambda, R)$ is Λ-projective

Also, Auslander–Buchsbaum holds for Λ-modules in these cases

<u>Defn:</u> A <u>non-commutative resolution of singularities</u> of a Gorenstein local ring R is an R-algebra Λ which is:

i) birational ($\Lambda \otimes_R K \cong \mathrm{Mat}_n(K)$)

ii) module-finite R-order

iii) non-singular

<u>Remarks:</u> 1) This is a non-standard definition

2) It makes sense for CM local ring R, but is less well-behaved.

– A very strong condition on usual resolution of singularities is that it is <u>crepant</u>: $\pi: Y \to X$ is crepant if $\pi^* \omega_X = \omega_Y$

Informally, induction and coinduction coincide for ω_X:

$$\mathcal{O}_Y \otimes_{\mathcal{O}_X} \omega_X \quad \text{vs.} \quad \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_Y, \omega_X) = \omega_Y$$

In particular, if $\omega_X = \mathcal{O}_X$, this says $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) = \mathcal{O}_Y$

<u>Defn:</u> An R-algebra Λ is <u>symmetric</u> if $\mathrm{Hom}_R(\Lambda, R) = \Lambda$ as bimodules.

<u>Defn:</u> A <u>non-comm. crepant resolution</u> of Gorenstein R is an R-order Λ which is non-singular and <u>symmetric</u>.

[Auslander '86]: If R is a normal domain, then $\mathrm{End}_R(M)$, M reflexive, is symmetric.

In fact,

<u>Theorem</u>: [Auslander - Goldman, Iyama - Reiten, Brown - Hajarnavis - Mactachean]: The following are equivalent for R Gorenstein and Λ module-finite:

1) Λ is a non-comm. crepant resolution of singularities
2) $\Lambda \cong \mathrm{End}_R(M)$, M reflexive, is non-singular and is R-order
3) $\Lambda \cong \mathrm{End}_R(M)$, M reflexive, and all simple Λ-modules have proj.dim. $=$ dim R ("homological homogeneity")

<u>Rmk</u>: 3) is the original defn. of Van den Bergh.

<u>Fact</u>:

If $\pi : Y \to X$ is a crepant res. of X Gorenstein, then X has rational singularities

<u>Theorem</u> [Stafford - Van den Bergh '08] If R is Gorenstein and has a non-comm. crepant res., then R has rational singularities.

- <u>Open Problems</u>:

1) <u>Conjecture</u> [Bondal - Orlov] Two crepant resolutions of singularities have equivalent $D^b(-)$'s.

↳ Van den Bergh '04: OK for dim $X = 3$, terminal singularities by showing $D^b(Y) \simeq D^b(\Lambda)$

(Bridgeland '04 also)

<u>Q</u>: Are <u>all</u> crepant resolutions, comm. and non-comm., derived equivalent?
