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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen Email/Phone: van.nguyen3@gmail.com

Speaker's Name: Ragnar-Olaf Buchweitz

Talk Title: Variations on Hochschild cohomology I

Date: 01/28/13 Time: 9:15 am / pm (circle one)

List 6-12 key words for the talk: Bar construction, separable, derived category, Hochschild cohomology, graded commutative, Atiyah-Chern character, K-projective

Please summarize the lecture in 5 or fewer sentances: Introduce the bar construction of A (assoc. ring with 1) over B (comm. ring), followed by the definition of the Hochschild cohomology of A, HH*(A). Discuss the relation between HH*(A) and $\mathrm{Ext}_{A^{ev}}(A,A)$ and their multiplicative structures. Introduce Quillen's model using derived category. There is a natural ring homomorphism $\delta: HH(A) \longrightarrow Z(D(A))$ called the Hochschild-Atiyah-Chern character.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

01/28/13 Ragnar-Olaf Buchweitz: "Variations on Hochschild cohomology I."
9:15 am

Outline:

1) The cast of characters
2) Comparison and Multiplicative structure
3) Atiyah - Chern characters

Throughout this talk:

A is a ring (associative with unit)
B is commutative ring
$f: A \to B$ is a ring homomorphism, $f(B) \subseteq$ center of A
Mod (A) : right A-modules

We have:

$$- \otimes_B A = f^* \uparrow \downarrow f_* \uparrow f' = \mathrm{Hom}_B(A, -)$$

between $\mathrm{Mod}(A)$ (top) and $\mathrm{Mod}(B)$ (bottom).

Whenever (L,R) is a pair of adjoint functors, we can form a complex:

$$\mathbb{B}: \quad id \xleftarrow[\text{counit}]{\eta} LR \underset{LR\eta}{\overset{\eta LR}{\leftleftarrows\!\!\rightarrow}} LRLR \quad \cdots \quad \underset{\partial_n}{\overset{\partial_1}{\rightleftarrows}} (LR)^n$$

$$L \circ id \circ R \xrightarrow[L\circ\text{unit}\circ R]{} L \circ R \circ L \circ R$$

Simplicial complex of functors $R\mathbb{B}$: is contractible.
Thus, if R defects exactness, then $\mathbb{B}$ is exact.

If (L, R) are functors between abelian categories, then can form 2 complexes:

$$(\mathbb{B}, \partial = \Sigma (-1)^i \partial_i) \equiv \mathrm{Im}(\sigma's) \text{ is contractible}$$
$$(N\mathbb{B}, \partial) = B/\mathrm{Im}(\sigma's)$$

Apply the construction to A: $L = f^*$, $R = f_*$

$\mathbb{B}(A)$, $N\mathbb{B}(A)$ are exact complexes of A-<u>bimodules</u>

let $M \in \mathrm{Mod}(A)$

$\varphi \in \mathrm{End}_A(M)$ acts from the left

$F(\varphi): F(M) \to F(M)$, $\mathrm{End}_A(A) \cong A$

$\mathbb{B}(A)$ is the <u>Bar</u> construction of A over B

$$(f^* f_*)^n (M) = M \otimes_B A^{\otimes n}$$

η : $M \otimes A \to M$: the module structure

∂_i : multiplication maps of adjacent factors

σ_i : inserting 1

<u>Defn</u>: A is <u>separable</u> over B if A is a direct summand of $A \otimes_B A$ as A-bimodule.

<u>Prop</u>: If $B \xrightarrow{g} B' \xrightarrow{f'} A$ are ring homomorphisms and if B' is separable over B, then $\mathbb{B}(A/B')$ is homotopy equivalent to $\mathbb{B}(A/B)$

<u>Note</u>: There is a natural morphism of complexes
$$\mathbb{B}(A/B) \longrightarrow \mathbb{B}(A/B')$$

ex: (Cibils)

let K be a comm. ring

Q be a quiver (finite)

KQ : path algebra, with Q_0 is the set of vertices

$$\begin{tikzcd} KQ \\ K^{Q_0} \arrow[u] \\ K \arrow[u, "\text{separable}"'] \end{tikzcd}$$

Following material can be found in books by Loday, Cauty - Quillen

Defn: Given an A-bimodule M,

$H^*(\mathrm{Hom}_{A-A}(\mathbb{B}A, M)) =: HH^*(A/B, M)$ is the Hochschild cohomology with coefficients in M, we denote $HH^*(A) = HH^*(A/B, A)$.

- Let B be commutative, A is a B-algebra

$\mathbb{B}(A) \longrightarrow A$ resolves

If $\mathcal{P}$ is a projective bimodule resolution, then we can lift $\mathcal{P}$ to $\mathbb{B}(A)$ as:

$$\begin{tikzcd} \mathbb{B}(A) \arrow[r] & A \\ \mathcal{P} \arrow[u, dashed] \arrow[ur] & \end{tikzcd}$$

$f^* f_* : \mathrm{Mod}\, A \to \mathrm{Mod}\, A$

let $A^{ev} = A^{op} \otimes_B A$ be the enveloping algebra

$HH^*(A) \longrightarrow \mathrm{Ext}_{A^{ev}}(A, A)$

If $\underline{A}$ is any abelian B-linear category, then we can define

$\underline{\mathrm{End}}_B(\underline{A})$: category of all B-linear endo. functors $F: \underline{A} \to \underline{A}$

: is abelian

$$\begin{tikzcd}
HH^*(A) \arrow[rr] \arrow[dr, "\alpha"'] & & \mathrm{Ext}_{A^{ev}}(A,A) \\
& \mathrm{Ext}_{\underline{\mathrm{End}}_B(A)}(id, id) \arrow[ur, "\beta"'] &
\end{tikzcd}$$

graded comm.: $HH^*(A)$

not always graded comm.: $\mathrm{Ext}_{A^{ev}}(A,A)$

graded comm.: $\mathrm{Ext}_{\underline{\mathrm{End}}_B(A)}(id,id)$

$\underline{\mathrm{End}}_B(A) := \underline{\mathrm{End}}_B(\text{Mod-}A)$

α, β are homomorphisms of graded rings

<u>Note</u>: $[0 \to id \to F_n \to \cdots \to F_1 \to id \to 0] \in \mathrm{Ext}^n_{\underline{\mathrm{End}}_B(\underline{A})}(id, id)$

$\downarrow ev_A$

$[0 \to A \to F_n(A) \to \cdots \to F_1(A) \to A \to 0] \in \mathrm{Ext}^n_{A^{ev}}(A,A)$

One can ask when α, β are isomorphisms?

↳ β is an isomorphism if A is <u>flat</u> over B.

α " " " " is <u>projective</u> over B.

When is $\mathbb{B}(A)$ K-(Spaltenstein)-projective?

<u>Defn.</u>: $C_\bullet$ is <u>K-projective</u>, if for any <u>acyclic</u> complex $D_\bullet$, $\mathrm{Hom}(C_\bullet, D_\bullet)$ is acyclic.

<u>Remark</u>: $\mathbb{B}(A)$ is an A^{ev}-projective resolution if A is proj. over B.

$N\mathbb{B}(A)$ " " " " if A/B is proj. over B.

But K-projectivity is the same for both.

<u>Prop</u>: If $\mathbb{B}(A)$ is K-projective, then $\beta\alpha$ is an isomorphism.

<u>Theorem</u>: $HH^*(A)$ is graded commutative under Yoneda product.

$$\mathrm{Hom}^\bullet_{A^{ev}}(\mathbb{B}(A), A) \xleftarrow{\simeq} \mathrm{End}^\bullet_{A^{ev}}(\mathbb{B}(A))$$

Yoneda / composition

Q: If A is a B-algebra, when is $\mathrm{Ext}^{\bullet}_{A^{ev}}(A,A)$ graded comm.?

Theorem: (Quillen) True, if $\mathrm{Tor}^B_i(A,A)=0$, $\forall i\neq 0$

ex: $B = k[x,y]/(y^2-x^3) \xhookrightarrow[y\mapsto t^3]{x\mapsto t^2} A = k[t]$

$\mathrm{Ext}_{A^{ev}}(A,A)$ is not graded commutative.

- Quillen models: (using derived category)

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr] & & A \\ & (\mathcal{A},\partial) \arrow[ur, "\varphi", "\sim"'] & \end{tikzcd}$$

$(\mathcal{A},\partial)$: free B DG-algebra

Fact: $D(A) \cong D(\mathcal{A})$

so we can form $\mathrm{Ext}_{\mathcal{A}^{ev}}(\mathcal{A},\mathcal{A}) =: \mathbb{HH}(A)$

so, $\mathrm{Ext}_{A^{ev}}(A,A) \xrightarrow{\varphi^*} \mathrm{Ext}_{\mathcal{A}^{ev}}(\mathcal{A},\mathcal{A})$

not always graded comm. graded comm.

$\mathcal{A}^{ev} = \mathcal{A}^{op}\otimes_B \mathcal{A} \implies H_\bullet(\mathcal{A}^{ev}) \cong \mathrm{Tor}^B(A,A)$

$$\begin{tikzcd} \mathcal{A}^{ev} \arrow[d, "\cong"', "\mathrm{Tor}_+(A,A)=0"] \\ A^{ev} \end{tikzcd}$$

Thus φ^* is an isomorphism.

- There is a natural homomorphism of graded commutative algebras

$\mathbb{HH}(A) = \mathrm{Ext}_{\mathcal{A}^{ev}}(\mathcal{A},\mathcal{A}) \xrightarrow[\delta]{} Z(D(A))$

$D(A)$ comes with a translation functor T

$Z^i(D(A)) := \{ z : \mathrm{id} \to T^i \mid zT = (-1)^i Tz \}$ is an algebra under composition

$D(A) \cong D(\mathcal{A})$

$Z(D(A)) \cong Z(D(\mathcal{A}))$

$\zeta \in \mathrm{Ext}^i_{\mathcal{A}^{ev}}(\mathcal{A}, \mathcal{A})$, $\zeta : \mathcal{A} \to T^i \mathcal{A}$ in $D(\mathcal{A}^{ev})$

$X \in D(\mathcal{A})$, $X \cong X \otimes_{\mathcal{A}} \mathcal{A} \xrightarrow[X \otimes \zeta]{} X \otimes_{\mathcal{A}} T^i(\mathcal{A}) \cong T^i X$

$\delta(\zeta) = - \otimes \zeta \in Z^i(D(A))$

<u>Defn</u>: δ is the <u>Atiyah-Chern character</u> of A over B.

(continued lecture II on 01/29/13)