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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Paul Smith

Talk Title: Introduction to non-commutative algebraic geometry I

Date: 01/28/13 **Time:** 2:00 am / (pm) (circle one)

List 6-12 key words for the talk: Noncommutative algebraic geometry (NCAG), Algebraic Noncommutative geometry (ANCG), affine NC-schemes, coherent sheaves

Please summarize the lecture in 5 or fewer sentances: Introduce noncommutative (NC) analogues of various projective surfaces, affine and projective NC-schemes, basic ideas and concepts in NCAG and link to ANCG.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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Noncommutative Algebraic Geometry

and

Algebraic Noncommutative Geometry

First Lecture

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Introductory Workshop:
Noncommutative Algebraic Geometry and Representation Theory

MSRI

Coordinate rings of finite T_0 topological spaces

$$\mathfrak{Cat}(\text{finite } T_0 \text{ topological spaces}) \equiv \mathfrak{Cat}(\text{finite posets})$$

$$y \in \overline{\{x\}} \quad \Longleftrightarrow \quad x \leq y$$

Proposition (Ladkani)

Let X be a finite T_0 topological space. Then

$$\mathfrak{Sh}\big(\mathbb{C}\text{-vector spaces on } X\big) \equiv \mathfrak{Mod}\big(\mathcal{O}(X)\big)$$

where $\mathcal{O}(X) := \operatorname{span}\{e_{xy} \mid x \leq y\}$ with multiplication

$$e_{wx}e_{yz} = \delta_{xy}e_{wz}.$$

$\mathcal{O}(X) =$ the incidence algebra of the poset $(X, \leq)$

Continuous maps $\longleftrightarrow$ bimodules

Proposition

Let $f : X \to Y$ be a continuous map between finite T_0 topological spaces. There is an $\mathcal{O}(X)$-$\mathcal{O}(Y)$-bimodule B_f such that

$$
\begin{array}{ccccc}
\mathfrak{Sh}(Y) & \xrightarrow{\quad f^{-1} \quad} & \mathfrak{Sh}(X) & \xrightarrow{\quad f_* \quad} & \mathfrak{Sh}(Y) \\
\equiv \big\downarrow & & \equiv \big\downarrow & & \equiv \big\downarrow \\
\mathfrak{Mod}(\mathcal{O}(Y)) & \xrightarrow[B_f \otimes -]{} & \mathfrak{Mod}(\mathcal{O}(X)) & \xrightarrow[\mathrm{Hom}(B_f, -)]{} & \mathfrak{Mod}(\mathcal{O}(Y))
\end{array}
$$

commutes.

- $(f^{-1}\mathcal{F})_x = \mathcal{F}_{f(x)}$
- $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U)$
- *If* $X \xrightarrow{f} Y \xrightarrow{g} Z$*, then* $B_{gf} \cong B_f \otimes_{\mathcal{O}(Y)} B_g$.

$\mathcal{O}(X)$ embeds in upper $\triangle^{\mathrm{ar}}$ matrices

Lemma

If $|X| = n$ there is an injective homomorphism

$$\mathcal{O}(X) \hookrightarrow \textit{upper } \triangle^{\mathrm{ar}} n \times n \textit{ matrices} \;\subset\; M_n(\mathbb{C}).$$

$$\text{e.g., } \mathcal{O}\big(\{1 < 2 < \cdots < n\}\big) \;\cong\; \text{upper } \triangle^{\mathrm{ar}}\; n \times n \text{ matrices.}$$

Idea of proof: Let $R(X) := \{(x, y) \mid x \le y\} \subset X \times X$. Then $\mathcal{O}(X) \cong \mathbb{C}^{R(X)} = \mathbb{C}$-valued functions on $R(X)$ with the convolution product

$$(fg)_{xz} = (f * g)(x, z) = \sum_y f(x, y)g(y, z) = \sum_y f_{xy} g_{yz}$$

Points = simple modules = skyscraper sheaves

$\mathfrak{m}_x := \{f \in \mathcal{O}(X) \mid f(x,x) = 0\}$
$\quad = $ maximal 2-sided ideal of $\mathcal{O}(X)$ of codimension 1.

Bijections:

points of $X \longleftrightarrow$ simple $\mathcal{O}(X)$-modules $\longleftrightarrow$ skyscraper sheaves
$x \longleftrightarrow \mathcal{O}(X)/\mathfrak{m}_x \longleftrightarrow \mathcal{O}_x$

Proposition (Stanley & Sorkin)

The following are equivalent :

1. $x \neq y$ *and* $[x,y] = \{x,y\}$
2. $\mathfrak{m}_x\mathfrak{m}_y \neq \mathfrak{m}_x \cap \mathfrak{m}_y$
3. $\mathrm{Ext}^1(\mathcal{O}_y, \mathcal{O}_x) \neq 0$
4. $\exists$ *s.e.s.* $0 \to \mathcal{O}_x \to M \to \mathcal{O}_y \to 0$ *with* $M \ncong \mathcal{O}_x \oplus \mathcal{O}_y$.

Corollary

Can recover X as a topological space from $\mathfrak{Mod}(\mathcal{O}(X))$.

$\mathcal{O}(X)$ is a coordinate ring for X

1. $\mathfrak{Mod}(\mathcal{O}(X)) \equiv \mathfrak{Sh}(X)$.
2. The structure map $\mathbb{C} \to \mathcal{O}(X)$ corresponds to the structure map $X \to \bullet = \operatorname{Spec} \mathbb{C}$.
3. $i : \{x\} \hookrightarrow X$ corresponds to $\mathfrak{m}_x \hookrightarrow \mathcal{O}(X) \twoheadrightarrow \mathcal{O}(\{x\})$.
4. Simple $\mathcal{O}(X)$-modules are the skyscraper sheaves at the points of x.
5. $\mathcal{O}(X \times Y) \cong \mathcal{O}(X) \otimes \mathcal{O}(Y)$.
6. $\mathcal{O}(X^{\mathrm{op}}) \cong \mathcal{O}(X)^{\mathrm{op}}$.
7. $X_{discrete} \to X$ corresponds to $\mathcal{O}(X) \twoheadrightarrow \mathcal{O}(X_{discrete}) = \mathcal{O}(X)/\sqrt{0}$ where $\sqrt{0} :=$ the largest nilpotent ideal in $\mathcal{O}(X)$.

In NCG points can talk to each other (Connes)

The essence of non-commutativity: $\mathrm{Ext}^1(\mathcal{O}_y, \mathcal{O}_x)$ can be non-zero when $\mathcal{O}_x$ and $\mathcal{O}_y$ are non-isomorphic simples.

Contrast with the commutative case:

Proposition

If X is a scheme, $\mathcal{M}, \mathcal{N} \in \mathrm{coh}(X)$, and $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(\mathcal{N}) = \varnothing$, then

$$\mathrm{Ext}^q_X(\mathcal{M}, \mathcal{N}) = 0 \quad \textit{for all } q \geq 0.$$

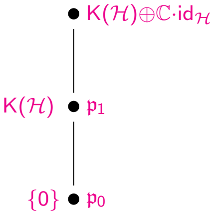
In particular, if $\mathcal{M}$ and $\mathcal{N}$ are non-isomorphic simples/skyscrapers, then every short exact sequence $0 \to \mathcal{N} \to \mathcal{F} \to \mathcal{M} \to 0$ splits.

The smallest non-commutative ring is $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix} = \mathcal{O}(\{0 < 1\})$.

The Sierpinski topological space, $\{0 < 1\}$, is a closed subspace of "most" non-commutative spaces.

Philosophy: nc C*-algs ↔ nc loc. comp. Hausdorff spaces

Up to Morita equivalence, the smallest non-comm. C*-algebra is



$K(\mathcal{H})$ = compact op'ors on an ∞-dim'l separable Hilbert space $\mathcal{H}$

$K(\mathcal{H})$ is strongly Morita equivalent to $\mathbb{C}$

$$\mathrm{Prim}\, K(\mathcal{H}) \cong \{0 < 1\}$$

Philosophy of NCAG or ANCG

A non-commutative variety or scheme X is made manifest by the category of modules or quasi-coherent sheaves that live on it,

$$\mathfrak{Qcoh}(X).$$

$\mathfrak{Qcoh}(X)$ holds contains algebraic and geometric information about X.

Theorem (Gabriel+Rosenberg)

A quasi-projective scheme X can be recovered from $\mathfrak{Qcoh}(X)$.

A nc-morphism $f : X \to Y$ between nc-schemes is
an adjoint pair of functors $f^* \dashv f_*$

$$\mathfrak{Qcoh}(X) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathfrak{Qcoh}(Y)$$

$f^* :=$ the inverse image functor $\qquad$ $f_* :=$ the direct image functor

f is affine if f_* is faithful.

$R =$ a ring $\qquad\qquad \mathfrak{Mod}(R) :=$ right R-modules

Define $\text{Spec}_{nc}(R)$ implicitly by declaring that

$$\mathfrak{Qcoh}\big(\text{Spec}_{nc}(R)\big) := \mathfrak{Mod}(R).$$

A ring homomorphism $\varphi : R \to S$ induces a nc-morphism

$$f : \text{Spec}_{nc}(S) \to \text{Spec}_{nc}(R)$$

$$\mathfrak{Mod}(S) \underset{f_*}{\longrightarrow} \mathfrak{Mod}(R), \qquad \mathfrak{Mod}(R) \xrightarrow{f^* \;\; f^!} \mathfrak{Mod}(S)$$

$$f^* \dashv f_* \dashv f^!$$

$$f^* = - \otimes_R S \qquad f_* = \text{Hom}_S(S, -) \qquad f^! = \text{Hom}_R(S, -)$$

Affine nc-schemes II: coordinate rings

A nc-scheme X is affine if there is a ring R such that

$$\mathfrak{Qcoh}(X) \equiv \mathfrak{Mod}(R).$$

Call such R a coordinate ring of X.

Equivalently, X is affine $\Longleftrightarrow$ $\mathfrak{Qcoh}(X)$ has a progenerator.

progenerator := a finitely generated projective generator

P a progen'or $\Longrightarrow$ $\operatorname{Hom}_X(P,-) : \mathfrak{Qcoh}(X) \xrightarrow{\equiv} \mathfrak{Mod}\big(\operatorname{End}_X(P)\big)$

Theorem (Serre, FAC)

A noetherian scheme X is affine $\Longleftrightarrow$ $H^q(X,\mathcal{F}) = 0$ for all $\mathcal{F} \in \mathfrak{Qcoh}(X)$ and all $q > 0$ $\Longleftrightarrow$ $\mathcal{O}_X$ is a progenerator in $\mathfrak{Qcoh}(X)$.

Proof: $H^q(X,-) = R^q\Gamma(X,-) = \operatorname{Ext}^q_X(\mathcal{O}_X,-)$.

Theorem (Eilenberg-Watts)

If $f^* : \mathfrak{Mod}(S) \to \mathfrak{Mod}(R)$ is a right-exact functor commuting with direct sums, then $\exists$ an R-S-bimodule B such that $f^* \cong - \otimes_R B$ and $f^* \dashv f_* := \mathrm{Hom}_S(B, -)$.

$$\mathfrak{Cat}(\text{affine nc-schemes}) := 2 - \mathfrak{Cat}(\text{rings \& morphisms} = \text{bimodules}).$$

$$R \xrightarrow{\;B\;} S \xrightarrow{\;A\;} T, \qquad R \xrightarrow{\;B \otimes_S A\;} T$$

$$\mathfrak{Mod}(R) \xrightarrow[f_B^*]{- \otimes_R B} \mathfrak{Mod}(S) \xrightarrow[f_A^*]{- \otimes_S A} \mathfrak{Mod}(T)$$

$$\mathrm{Spec}_{nc}(R) \xleftarrow{\;f_B\;} \mathrm{Spec}_{nc}(S) \xleftarrow{\;f_A\;} \mathrm{Spec}_{nc}(T)$$

Some affine curves I

Let R be a fin. gend. comm. algebra over $k = \overline{k}$

Tautology:

- Spec $\big(R[t]\big) \cong \mathbb{A}^1_k \times \operatorname{Spec}(R) \xrightarrow{\xi} \operatorname{Spec}(R)$
- $\xi^{-1}(p) \cong \mathbb{A}^1_k$ for all closed points $p \in \operatorname{Spec} R$
- Spec $R[t]$ = the disjoint union of the fibers $\xi^{-1}(p)$

A non-commutative analogue:

Replace $R[t]$ by an Ore extension $A = R[t; \sigma, \delta]$ where $\sigma \in \operatorname{Aut}_k(R)$ and $\delta : R \to R$ is a k-linear map s.t. $\delta(rs) = \delta(r)s + \sigma(r)\delta(s)$ for all $r, s \in R$.

$$R[t; \sigma, \delta] = R \oplus Rt \oplus Rt^2 \oplus \cdots \qquad \text{where } tr = \sigma(r)t + \delta(r)$$

$R \hookrightarrow A$ induces an affine nc-morphism $\xi : X = \operatorname{Spec}_{nc}(A) \to \operatorname{Spec}(R)$.

What do the fibers $X_p = \xi^{-1}(p)$ look like?

The fibers should be considered as non-commutative curves

The fibers of ξ

Theorem (S-Zhang)

The fibers X_p have the following structure:

1. $p = \sigma(p)$ *and* $f(\delta)(R) \subset \mathfrak{m}_p$ *for some* $f \in k[t] - \{0\} \Longrightarrow X_p \cong \mathbb{A}^1_k$
2. $p = \sigma(p)$ *and case (1) does not occur* $\Longrightarrow X_p \cong \operatorname{Spec}(k)$
3. $|\sigma^{\mathbb{Z}}(p)| = n < \infty \Longrightarrow \mathfrak{Qcoh}(X_p) \equiv \mathfrak{Mod}(kQ)$ *where* $Q = \widetilde{A_n}$ *with cyclic orientation*
4. $|\sigma^{\mathbb{Z}}(p)| = \infty \Longrightarrow \mathfrak{Qcoh}(X_p) \equiv \mathfrak{Gr}(k[u]) \equiv \mathfrak{Mod}(\mathcal{O}(\mathbb{Z}, \leq))$ *with* $\deg(u) = 1$

k *uncountable* $\Longrightarrow X$ *is the disjoint union of the fibers.*

How is X_p defined?

The definition of $\mathfrak{Qcoh}(X_p)$ involves the injective envelope

$$E(\xi^* \mathcal{O}_p) = E(A/\mathfrak{m}_p A).$$

Exceptional locus for blowing up a point on a nc-surface

- X = noetherian nc surface
- $Y \subset X$ is a comm. curve that is a divisor & $y \in Y$ is a closed point
- $\text{inj.dim}_{\mathfrak{Qcoh}(X)} \mathcal{F} < \infty \;\; \forall \; \mathcal{F} \in \mathfrak{Qcoh}(Y)$ (X smooth in a nghd of Y)

Theorem (Van den Bergh)

There is a comm. diagram

$$\begin{array}{ccccc} \pi^{-1}(y) = E & & \tilde{Y} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow & & \downarrow \pi \\ \{y\} & \longrightarrow & Y & \longrightarrow & X \end{array} \qquad (E \longrightarrow \tilde{X})$$

in which

1. $\widetilde{X} - E \cong X - \{y\}$
2. $\widetilde{Y}$ = *"strict transform" of Y is commutative curve & a divisor on $\widetilde{X}$*
3. $\sigma(y) = y \Longrightarrow E \cong \mathbb{P}^1$, *i.e.*, $\mathfrak{Qcoh}(E) \equiv \mathfrak{Qcoh}(\mathbb{P}^1)$
4. $|\sigma^{\mathbb{Z}}(y)| = n \Longrightarrow E \cong \mathbb{P}^1_{[1,n]} \cong [\mathbb{P}^1/\mu_n]$ *i.e.*, $\mathfrak{Qcoh}(E) \equiv \mathfrak{QGr}(k[u,v])$ *with* $\deg(u,v) = (1,n)$
5. $|\sigma^{\mathbb{Z}}(y)| = \infty \Longrightarrow \mathfrak{Qcoh}(E) \equiv \mathfrak{Gr}(k[v])$ *with* $\deg(v) = 1$

$\sigma(y) = y \iff \pi^{-1}(y) \cong \mathbb{P}^1$
$\sigma(p) = p \Longleftarrow \xi^{-1}(p) \cong \mathbb{A}^1 = \mathbb{P}^1 - \{\text{point}\}$

$|\sigma^{\mathbb{Z}}(y)| = n < \infty \iff \pi^{-1}(y) \cong [\mathbb{P}^1/\mu_n]$
$|\sigma^{\mathbb{Z}}(p)| = n < \infty \iff \xi^{-1}(p) \cong [\mathbb{A}^1/\mu_n] = [\mathbb{P}^1/\mu_n] - \{\text{point}\}$

$|\sigma^{\mathbb{Z}}(y)| = \infty \iff \pi^{-1}(y) \cong (\mathbb{Z}, \leq)$
$|\sigma^{\mathbb{Z}}(p)| = \infty \iff \xi^{-1}(p) \cong (\mathbb{Z}, \leq) = (\mathbb{Z}, \leq) - \{\text{point}\}$

Use a quotient category to formalize $X_{nc} - \{\text{a closed point}\} \quad \hookrightarrow \quad X_{nc}$

Conclusion: the above examples are typical affine and projective nc curves

BUT there is one strange feature: the fiber $\xi^{-1}(p) \cong \text{Spec}(k)$

BUT some points on non-comm. surfaces behave like curves with negative self-intersection

OR some curves on non-comm. surfaces behave like points

$$\mathfrak{sl}(2,\mathbb{C}) = \operatorname{span}\left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

V_n := irreducible representation of dimension $n+1$.

$\Omega := 2(ef + fe) + h^2$, the Casimir central element

$$U_\lambda := \frac{U(\mathfrak{sl}(2,\mathbb{C}))}{(\Omega - \lambda)}$$

$Q_\lambda := \text{Spec}_{nc}(U_\lambda) \subset \text{Spec}_{nc}\left(U(\mathfrak{sl}(2,\mathbb{C}))\right)$
A pencil of nc-quadric surfaces in an $\mathbb{A}^3_{\mathrm{nc}}$.
Analogous to the conjugacy classes $det = \lambda$ in $\mathfrak{sl}(2,\mathbb{C})$.

The pencil Q_λ has the "same" singularity behavior as the commutative pencil of quadrics $x^2 + y^2 + z^2 = \lambda$

Proposition (Stafford)

$$\operatorname{gldim}(U_\lambda) = \begin{cases} \infty & \textit{if } \lambda = -1 \\ 2 & \textit{if } \lambda = n(n+2) \textit{ for some } n \in \mathbb{N} \\ 1 & \textit{otherwise} \end{cases}$$

Proposition (Van den Bergh)

$$\operatorname{pdim}_{U_\lambda^e}(U_\lambda) = \begin{cases} \infty & \textit{if } \lambda = -1 \\ 2 & \textit{otherwise} \end{cases}$$

(Twisted) Hochschild cohomology dimension is a "better" measure of dimension that global dimension (when it is finite).

$\mathfrak{m}_n := \text{Ann}(V_n) \supset (\Omega - n(n+2))$

$$\frac{U(\mathfrak{sl}(2,\mathbb{C}))}{\mathfrak{m}_n} \cong M_{n+1}(\mathbb{C}) \overset{M.E.}{\sim} \mathbb{C}$$

The zero-locus of $\mathfrak{m}_n$ is $Z(\mathfrak{m}_n) := \text{Spec}_{nc}\left(U_{n(n+2)}/\mathfrak{m}_n\right) \cong \text{Spec}(\mathbb{C})$

$\mathcal{D}$-modules $=$ quasi-coh. sheaves of modules over the sheaf of diff'l ops.

- $U_0 \cong \Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1})$ and $\mathfrak{Mod}(U_0) \equiv \mathfrak{Qcoh}(\mathcal{D}_{\mathbb{P}^1})$
- Under this equivalence, $V_0 \longleftrightarrow \mathcal{O}_{\mathbb{P}^1}$, an object of "geom. dim. 1".
- Although $Z(\mathfrak{m}_n)$ has "dimension zero" for several reasons, from the perspective of $\mathcal{D}$-modules it has dimension 1.

Strange points on $Q_{n(n+2)}$, continued

If p is a closed point on a smooth commutative quadric, then

$$\dim_k \operatorname{Ext}^0(\mathcal{O}_p,\mathcal{O}_p) - \dim_k \operatorname{Ext}^1(\mathcal{O}_p,\mathcal{O}_p) + \dim_k \operatorname{Ext}^2(\mathcal{O}_p,\mathcal{O}_p) = 1-2+1 = 0$$

however, on $Q_{n(n+2)}$,

$$\dim_k \operatorname{Ext}^0(V_n,V_n) - \dim_k \operatorname{Ext}^1(V_n,V_n) + \dim_k \operatorname{Ext}^2(V_n,V_n) = 1-0+1 = 2$$

There are other ways the point $Z(\mathfrak{m}_n)$ behaves like a curve.

- There is a pencil of nc projective quadrics $\overline{Q_\lambda} \subset \mathbb{P}^3_{\mathrm{nc}}$, and
- an intersection theory on $\overline{Q_\lambda}$ for $\lambda \neq -1$
- The "point" on $\overline{Q}_{n(n+2)}$ corresponding to $Z(\mathfrak{m}_n)$ has self-intersection number 2
- $\overline{Q}_\lambda$ is ruled by two pencils of lines.
- Each pencil is naturally parametrized by {Borel subalgebras}.
- The lines in each pencil correspond to the Verma modules $\mathbb{C}_{n^2 \pm 2n} \otimes_{U(\mathfrak{b})} U\big(\mathfrak{sl}(2,\mathbb{C})\big)$ as $\mathfrak{b}$ ranges over all Borel subalgebras.
- The point $Z(\mathfrak{m}_n)$ lies on all the lines in one of the rulings.

HENCEFORTH $A = K \oplus A_1 \oplus A_2 \oplus \cdots$ is

- a finitely generated k-algebra (k=field) &
- locally finite, i.e., $\dim_k(A_n) < \infty$ for all n.

e.g. $A = kQ/I$, a quotient of the path algebra of a finite quiver Q.

Enter the hero: $\mathfrak{QGr}(A)$

Theorem (Serre, 1955, FAC, §59)

If $S = k[x_0, \ldots, x_n]$ is the polynomial ring on $n+1$ variables with $\deg(x_i) = 1$ and I is an ideal generated by homogeneous elements, then

$$\mathfrak{QGr}(S/I) \equiv \mathfrak{Qcoh}\big(\operatorname{Proj}(S/I)\big),$$

the cat. of quasi-coherent sheaves on the projective scheme $\operatorname{Proj}(S/I)$.

$$\text{e.g. } \mathfrak{QGr}(S) \equiv \mathfrak{Qcoh}(\mathbb{P}^n)$$

$\mathfrak{QGr}(A)$ can be defined for any graded ring A

k = a field & A = a $\mathbb{Z}$-graded k-algebra.

Abelian categories:

$\mathfrak{Gr}(A)$:= $\mathbb{Z}$-graded right A-modules with degree-preserving A-module homomorphisms

$\mathfrak{Fdim}(A)$:= the full subcategory of $\mathfrak{Gr}(A)$ consisting of the M that are the sum of their finite dim'l submodules

Because $\mathfrak{Fdim}(A)$ is closed under submodules, quotients, and extensions, there is a quotient category

$$\mathfrak{QGr}(A) := \frac{\mathfrak{Gr}(A)}{\mathfrak{Fdim}(A)}$$

Define $\mathrm{Proj}_{nc}(A)$ implicitly by declaring that

$$\mathfrak{Qcoh}(\mathrm{Proj}_{nc}(A)) := \mathfrak{QGr}(A).$$

Call A a homogeneous coordinate ring (hcr) for $\mathrm{Proj}_{nc}(A)$

Concerning $\mathfrak{QGr}(A)$

- $\text{Ob}(\mathfrak{QGr}(A)) = \text{Ob}(\mathfrak{Gr}(A))$ but $\mathfrak{QGr}(A)$ has more morphisms
- There is an exact localization functor $\pi^* : \mathfrak{Gr}(A) \to \mathfrak{QGr}(A)$.
- π^* has an exact right adjoint, π_*.
- $M \in \mathfrak{Fdim}(A) \iff \pi^* M \cong 0$.
- Twisting: if $M \in \mathfrak{Gr}(A)$ and $n \in \mathbb{Z}$, define $M(n) \in \mathfrak{Gr}(A)$ by $M(n)_A = M_A$ but $M(n)_i := M_{n+i}$.
- $M \mapsto M(n)$ is an automorphism of $\mathfrak{Gr}(A)$ and $\mathfrak{Fdim}(A)$ so induces an automorphism $(n) : \mathfrak{QGr}(A) \to \mathfrak{QGr}(A)$
- Write $X = \text{Proj}_{nc}(A)$.
- Often write $\mathcal{O}_X$ for $\pi^* A$ and consider the pair $(X, \mathcal{O}_X)$.
- Call $\mathcal{O}_X$ a structure sheaf for X.

The above facts are compatible with Serre's Theorem:

- If $M \in \mathfrak{Gr}(S/I)$, then $\pi^*(M) \cong \widetilde{M}$ à la Hartshorne pp. 116-117.
- $\pi^*(S/I) \cong \mathcal{O}_{\text{Proj}(S/I)}$
- (n) = the usual Serre twist/degree shift.

If A is a right noetherian graded algebra define

- $\mathfrak{gr}(A)$:= finitely generated graded A-modules $\subset \mathfrak{Gr}(A)$
- $\mathfrak{fdim}(A)$:= finite dim'l graded A-modules $= \mathfrak{gr}(A) \cap \mathfrak{Fdim}(A)$

$$\mathfrak{qgr}(A) := \frac{\mathfrak{gr}(A)}{\mathfrak{fdim}(A)} \subset \mathfrak{QGr}(A) \tag{1}$$

In Serre's theorem, $\mathfrak{qgr}(S/I) = \mathfrak{coh}(\operatorname{Proj}(S/I))$

If A is a graded algebra that is right graded-coherent define

- $\mathfrak{gr}(A)$:= finitely presented graded right A-modules $\subset \mathfrak{Gr}(A)$
- $\mathfrak{fdim}(A)$:= finitely presented finite dim'l graded right A-modules $= \mathfrak{gr}(A) \cap \mathfrak{Fdim}(A)$
- Define $\mathfrak{qgr}(A)$ by (1) above

A ring R is right coherent if every finitely generated submodule of a finitely presented module is finitely presented. Equivalently, the category $\operatorname{mod}(R)$ of finitely presented right R-modules is abelian.

The path algebra, kQ, of every finite quiver is coherent

A nc hcr for $\mathbb{P}^1$

If $A = \dfrac{k\langle x,y\rangle}{(yx - xy - x^2)}$ *OR* $A = \dfrac{k\langle x,y\rangle}{(yx - qxy)}$ for some $q \in k^\times$, then

$$\mathfrak{QGr}(A) \equiv \mathfrak{Qcoh}(\mathbb{P}^1).$$

Reason: A is a Zhang twist of the polynomial ring so

$$\mathfrak{Gr}(A) \equiv \mathfrak{Gr}\big(k[X,Y]\big).$$

How is $\mathbb{P}^1$ obtained from A?

Answer: $\mathbb{P}^1$ = the moduli space for the point modules for A

A point module for a conn. gr. k-alg. $A = k[A_1]$ is a graded right A-module

$$M = \bigoplus_{n=0}^{\infty} M_n \quad s.t. \quad M_n = M_0 A_n \ \& \ \dim_k(M_n) = 1 \ \forall\, n \geq 0.$$

The point modules for the above A are

$$M_p := \frac{A}{(bx - ay)A}, \qquad \text{parametrized by } p = (a,b) \in \mathbb{P}^1.$$

A nc hcr for $\mathbb{P}^1 \times \mathbb{P}^1$

$$B := \frac{k\langle x, y\rangle}{[x^2, y] = [y^2, x] = 0} = \frac{k\langle x, y\rangle}{(x^2y - yx^2, xy^2 - y^2x)}$$

$$\mathfrak{QGr}(B) \equiv \mathfrak{Qcoh}(\mathbb{P}^1 \times \mathbb{P}^1)$$

Reason: Verevkin's Theorem OR Artin-Van den Bergh Theorem for twisted hcrs.

- B is 3-dim'l AS-regular & $H(B;t) = (1-t^2)^{-1}(1-t)^{-2}$
- $B^{(2)} = k[B_2] = k[x^2, xy, yx, y^2]$
- $B^{(2)}$ is commutative & $\cong k[x_0, x_1, x_2, x_3]/(x_0x_3 - x_1x_2)$
- Proj $\left(B^{(2)}\right) =$ (a smooth quadric in $\mathbb{P}^3$) $\cong \mathbb{P}^1 \times \mathbb{P}^1$

Veronese subalgebras & Verevkin's Theorem

Def'n. Let A be an $\mathbb{N}$-graded k-algebra. Call

$$A^{(n)} := \bigoplus_{i=0}^{\infty} A_{in}$$

with $\deg(A_{in}) = i$, the n-Veronese subagebra of A.

Theorem (Verevkin)

If $A = A_0[A_1]$ is coherent, then $\mathfrak{QGr}(A) \equiv \mathfrak{QGr}(A^{(n)})$ via $\pi^* M \rightsquigarrow \pi^*\left(M^{(n)}\right)$.

On the previous slide, $\operatorname{Proj}_{nc}(B) \cong \operatorname{Proj}_{nc}\left(B^{(2)}\right) \cong \mathbb{P}^1 \times \mathbb{P}^1$
i.e., $\mathfrak{QGr}(B) \equiv \mathfrak{QGr}\left(B^{(2)}\right) \equiv \mathfrak{Qcoh}(\mathbb{P}^1 \times \mathbb{P}^1)$

V's Thm. fails if $A \neq A_0[A_1]$. E.g., if $\deg(x) = r$, then

$$\mathfrak{QGr}(k[x]) \equiv \mathfrak{Mod}\left(k^{\oplus r}\right) \text{ but } \mathfrak{QGr}(k[x]^{(r)}) \equiv \mathfrak{Mod}(k).$$