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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Ragnar-Olaf Buchweitz

Talk Title: Variations on Hochschild cohomology II

Date: 01/29/13 **Time:** 9:15 am / pm (circle one)

List 6-12 key words for the talk: syzygy, Atiyah class, Atiyah-Chern character, hereditary, Hochschild-Mitchell cohomology

Please summarize the lecture in 5 or fewer sentances: correction to the definition of the Atiyah-Chern character. Discuss the stabilized version of this ring homomorphism into the singularity category of a Gorenstein singularity. Introduce Hochschild-Mitchell cohomology and compare it with the usual Hochschild cohomology (Morita theorem).

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
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- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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01/29/13
9:15am

Ragnar-Olaf Buchweitz :
"Variations on Hochschild cohomology II"

- Throughout this talk, assume
  A: associative
  K: commutative
  $K \to A$ is an algebra
  $HH^0(A/K, M) = M^A = \{m \in M \mid am = ma\}$ invariants
  M is an A-bimodule which is K-symmetric
  $HH^1(A/K, M) \cong \mathrm{Der}_K(A, M)/\mathrm{Inn}(A, M)$

- Fundamental exact sequence:
$$0 \leftarrow A \underset{\text{mult.}}{\xleftarrow{\mu}} A^{op} \otimes_K A \longleftarrow \widetilde{\Omega}^1_{A/K} \longleftarrow 0$$

Fact: $\widetilde{\Omega}^1_{A/K}$ represents the functor
$$\text{Mod-}A^{op} \otimes_K A \ni M \longmapsto \mathrm{Der}_K(A, M) \cong \mathrm{Hom}_{A^{ev}}(\widetilde{\Omega}^1_{A/K}, M)$$
universal derivation
$$d : A \longrightarrow \widetilde{\Omega}^1_{A/K} \subseteq A \otimes_K A$$

$$d : \overline{A} := A/K \longrightarrow \widetilde{\Omega}^1_{A/K} \text{ sends } a \longmapsto 1 \otimes a - a \otimes 1 =: da$$

ex: $A = k[x_1, \ldots, x_n]$ polynomial ring
$$HH^*(A/K) \cong \mathrm{Ext}^*_{A \otimes_K A}(A, A)$$

$$A \otimes A \cong k[x_1', \ldots, x_n' ; x_1'', \ldots, x_n''] \xrightarrow[x_i'' \mapsto x_i]{x_i' \mapsto x_i} k[x_1, \ldots, x_n]$$

$$\widetilde{\Omega}^1_{A/K} = (x_i'' - x_i' \; ; i) \subseteq A \otimes A$$

$\Rightarrow$ Koszul complex $\mathbb{K}(x_i'' - x_i' ; A \otimes A)$ resolves A.

$\Rightarrow$ $\mathrm{Hom}_{A \otimes A}(\mathbb{K}, A) = 0 \leftarrow A \xleftarrow{0} \underset{\theta_A}{\overset{\oplus A}{\|}} \xleftarrow{\partial} \cdots \xleftarrow{\partial} \Lambda^i_A(\theta_A) \leftarrow 0$

<u>Theorem</u>: (Hochschild - Kostant - Rosenberg ; Loday - Quillen)

If A/k is commutative and smooth affine, then
$HH^i(A) \cong \Lambda^i_A \, \theta_A$: altternating "poly vector fields" of degree i
as A-modules

— <u>Caitz - Quillen 1995</u>: $\rho : k \to A$

The fundamental sequence splits as A-mod.

$$0 \leftarrow k \otimes_k A \underset{\rho \otimes i}{\xleftarrow{\mu}} A \otimes_k A \longleftarrow \bar{A} \otimes A \leftarrow 0$$

$1 \otimes a \mapsto 1 \otimes a$

<u>Corollary</u>: $\tilde{\Omega}^1_{A/k} \cong \bar{A} \otimes_k A \ni dx \otimes y$

$a\,dx \otimes y = d(ax) \otimes y - da\,x \otimes y$

$$0 \leftarrow A \leftarrow A \otimes_k A \leftarrow \tilde{\Omega}^1_{A/k} \leftarrow 0$$

tensor with $\tilde{\Omega}^{\otimes i}_{A/k} =: \tilde{\Omega}^i_{A/k}$, we get

$$0 \leftarrow \tilde{\Omega}^{\otimes i}_{A/k} \leftarrow A \otimes_k \tilde{\Omega}^i_{A/k} \leftarrow \tilde{\Omega}^{i+1}_{A/k} \leftarrow 0$$

$$A \otimes_k \tilde{\Omega}^i_{A/k} \cong A \otimes \bar{A}^{\otimes i} \otimes A$$

Concaternating, you get the normalized Bar construction.

Van den Bergh: A/k is <u>smooth</u> if A is flat over k and
$\mathrm{proj.dim.}_{A^{ev}} A < \infty \iff \tilde{\Omega}^i_{A/k}$ is proj. for some i over A^{ev}

<u>Corollary</u>: $\tilde{\Omega}^i_{A/k}$ is the i^{th} syzygy module of A/A^{ev}

$$HH^i(A) = \underbrace{\mathrm{Hom}_{A^{ev}}(\widetilde{\Omega}^i_{A/K}, A)}_{\text{poly vector fields of degree } i} \Big/ \text{"Inn vector fields"}$$

- From previous lecture (01/28/13), the definition of the Atiyah-Chern character was incorrect. We give the correction here.

If M is any A-module, then $M \otimes_A NB(A)$ is a resolution of M.
$\Rightarrow$ The i^{th} syzygy in this complex is $M \otimes_A \widetilde{\Omega}^i_{A/K}$
$\rightarrow$ Canonical class in $\mathrm{Ext}^i_A(M, M \otimes \widetilde{\Omega}^i_{A/K})$ is called the i^{th} Atiyah class of M.

Defn: The Atiyah-Chern character of an $X \in D(A)$ is the element

$$ch_{X/A} := (at^i_{X/A}) \in \prod_i \mathrm{Ext}^i_A(X, X \otimes^{\mathbb{L}}_A \widetilde{\Omega}^i_{A/K})$$

then

$$\begin{array}{ccccc} \delta : \mathrm{Ext}_{A^{ev}}(A,A) & \longrightarrow & Z(D(A)) & \xrightarrow{ev_X} & \mathrm{Ext}_A(X,X) \\ \wr\| & & & & \| \\ \dfrac{\mathrm{Hom}(\widetilde{\Omega}^{\bullet}_{A/K})}{\mathrm{Inn}} & & \xrightarrow{\quad - \lrcorner\, ch_{X/A} \quad} & & \mathrm{Ext}_A(X,X) \\ \quad\quad \theta & & \longmapsto & & \theta \lrcorner\, ch_{X/A} \end{array}$$

(A is associative, projective $/ K$)

- (Cartan) Connections:

let M be a right A-module.
A connection $\nabla : M \to M \otimes_A \widetilde{\Omega}^1_{A/K}$ satisfies K-linear and
$\nabla(ma) = \nabla(m)a + m \otimes da$

Fact: Such connection exists $\Leftrightarrow$ M is projective

If $M \cong A^{\oplus n}$ (free over A), we have Levi-Civita connection

$$\nabla : (a_1, \dots, a_n) \longmapsto (da_1, \dots, da_n) \in A^{\oplus n} \otimes_A \widetilde{\Omega}^1_{A/K}$$

Given arbitrary M, choose a projective resolution

$$0 \leftarrow M \leftarrow P_0 \xleftarrow{\partial} P_1 \xleftarrow{\partial} \dots \xleftarrow{\partial} P_i \xleftarrow{\partial} \dots$$

Then choose connections ∇_i on each P_i

<u>Exercise</u>: $[\partial, \nabla] : M \longrightarrow M \otimes \widetilde{\Omega}^1_{A/K}[1]$

<u>Claim</u>: $[\partial, \nabla]$ represents the 1^{st} Atiyah-Chern character of M in $D(A)$

The Atiyah-Chern character is represented by $([\partial, \nabla]^i)_i$

<u>Ex</u>: Assume $P_i \cong A^{\oplus n_i}$

The differentials can be represented by some matrix $(a_{rs})_{rs}$

$$\begin{tikzcd}
\partial : A^{\oplus n_i} \arrow[d, "{[\partial,\nabla] = (da_{pq})}"'] & P_{i+1} \cong A^{\oplus n_{i+1}} \arrow[l, "(a_{rs})_{rs}"'] \arrow[d, "{[\partial,\nabla] = (da_{rs})_{rs}}"] \\
P_{i-1} \cong A^{\oplus n_{i-1}} & P_i \otimes \widetilde{\Omega}^1_{A/K} \arrow[l, "-(a_{pq})"'] \arrow[d, "da_{pq} \otimes da_{rs}"] \\
 & P_{i-1} \otimes \widetilde{\Omega}^2_{A/K}
\end{tikzcd}$$

<u>Aside</u>: If M is perfect then we can take the trace

$$X \longrightarrow X \otimes \widetilde{\Omega}^i_{A/K}[i]$$

$$A \longrightarrow \mathrm{End}(X) \otimes \widetilde{\Omega}^i_{A/K}[i]$$

$$\mathrm{tr} : (A \longrightarrow \widetilde{\Omega}^i_{A/K}[i]) \ni ch^i_{X/A}$$

<u>Examples</u>: $A = k[x]$ or $A = k[x]_{(x)}$

$$HH^0(A) = A$$

$$HH^1(A) = A\frac{\partial}{\partial x}, \qquad HH^i(A) = 0, \ \forall i \geq 2$$

Consider

$$HH^*(A) \longrightarrow Z(D^b(A))$$

<u>Note</u>: $D^b(A) := D^b(\text{mod } A)$

A hereditary $\Rightarrow$ every complex is <u>formal</u> $:= C^\bullet \cong \bigoplus_i H^i(C^\bullet)[i]$ for $C^\bullet \in D^b(A)$

$\Updownarrow$

$\forall X, Y \in \text{Mod } A$, $\text{Ext}^i_A(X,Y) = 0$, $\forall i \geq 2$

(i.e. global dim $= 1$)

Now,

let $M \in \text{Mod } A$, $M \cong$ free $\oplus$ torsion, and the indecomposable torsion module

$$M_{a,n} := \frac{k[x]}{(x-a)^n} \quad \text{over } k[x]$$

$$\text{or } \frac{k[x]_{(x)}}{(x)^n} \quad \text{over } k[x]_{(x)}, \quad \text{where } k = \overline{k}$$

$$Z(D^b(A)): \quad Z^0 = A$$

$$Z^1(D^b(A)) = \prod_{a,n} \underbrace{\text{soc}(\text{Ext}^1_A(M_{a,n}, M_{a,n}))_{a,n}}_{\text{represent the Auslander-Reiten sequences (triangles)}}$$

$$z: id \to T$$

So now, the map $HH^1(A) \xrightarrow{\delta} Z^1(D^b(A))$

For $A = k[x]_{(x)}$: $\quad p\cdot\frac{\partial}{\partial x} \longmapsto \prod_{n,a} \text{Ext}^1_A(M_{n,a}, M_{n,a})$

$$\begin{array}{ccccccccc}
 & & 0 \to & k[x] & \xrightarrow{x^n} & k[x] & \longrightarrow & M_{n,0} & \to 0 \\
 & & & & & \downarrow dx^n = nx^{n-1}dx & & & \\
0 \to & k[x]dx & \xrightarrow{0} & & & k[x]dx & \to 0 & & \\
 & & & & & \downarrow p\cdot\frac{\partial}{\partial x} & & & \\
 & & & & & M_{n,0} & & &
\end{array}$$

so $\operatorname{Ker} \delta = (x)\frac{\partial}{\partial x}$

In contrast, for $A = k[x]$, δ is <u>injective</u>.

This generalizes: If A is any smooth affine scheme (finite type) over a field, then δ is injective.
(Grothendieck residues à la Lipman)

- <u>Hochschild-Mitchell cohomology</u>:
  let C be a small k-category
  $\operatorname{Mod} C := \{ F : C \to \operatorname{Mod} k \mid F \text{ is } k\text{-linear} \}$
  e.g. $\mathbb{B}A = \{ \cdot, \operatorname{End}(\cdot) = A \}$
  C_{dis} : remove all morphisms except scalar multiples of the identities
  $C_{dis} \subseteq C$; $\operatorname{Mod} C \underset{f^*}{\overset{f_*}{\rightleftarrows}} \operatorname{Mod} C_{dis}$

Define the Hochschild-Mitchell cohomology, A is any k-algebra
$HM(A) :=$ cohomology obtained from $(f^* f_*)$
with values in $\bigoplus_{X,Y} \operatorname{Hom}_C(X, Y)$
applied to $\operatorname{Mod} A =: C$

$HM(A)$ is not exactly the Hochschild cohomology of A, one can think of it as $HM(A)$ "=" $HH^*(\operatorname{Mod} A)$

$$HH(A) = HM(\mathbb{B}A) \cong HM(\underbrace{\operatorname{proj} A}_{\text{f.g. projective } A\text{-modules}})$$

<u>Morita Theorem</u>: If C, C' have the same additive closure, then
$$HM(C) \cong HM(C')$$

<u>Remark</u>: If A is representation-finite, then
$$\mathbb{HM}(\mathrm{mod}\,A) \cong \mathbb{HH}^*(\mathrm{Aus}(A))$$
where $\mathrm{Aus}(A) = \mathrm{End}(\bigoplus_i M_i)$ ← representative of indecomposable classes

- let A/K be any assoc. algebra, A is projective over K
$\mathbb{HM}^0(A) \cong \mathbb{HH}^0(A) \cong$ centre of A
$\exists$ exact sequence

$$0 \to \mathbb{HM}^1(A) \xrightarrow[\text{restriction}]{} \mathbb{HH}^1(A) \xrightarrow[\cong]{\alpha} \mathrm{Ext}^1_{\underline{\mathrm{End}}_K(\mathrm{Mod}\,A)}(\mathrm{id},\mathrm{id}) \xrightarrow{\delta}$$

$$Z^1(DA) \subseteq \prod_{X \in D(A)} \mathrm{Ext}^1_A(X,X) \to \mathbb{HM}^2(A) \to \mathrm{Ext}^2_{\underline{\mathrm{End}}_K(\mathrm{Mod}\,A)}(\mathrm{id}_A,\mathrm{id}_A)$$

$$\mathrm{Ext}^2_{\underline{\mathrm{End}}_K(\mathrm{Mod}\,A)}(\mathrm{id}_A,\mathrm{id}_A) = \mathbb{HH}^2(A)$$

If A is hereditary,
$$Z^1(DA) = \prod_{X \in D(A)} \mathrm{Ext}^1_A(X,X), \quad \text{and}$$

$$\mathrm{Ext}^2_{\underline{\mathrm{End}}_K(\mathrm{Mod}\,A)}(\mathrm{id}_A,\mathrm{id}_A) = 0$$
