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# NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen Email/Phone: van.nguyen3@gmail.com

Speaker's Name: Amnon Yekutieli

Talk Title: Introduction to Derived Categories II

Date: 01 / 29 / 13 Time: 11 : 00 (am) / pm (circle one)

List 6-12 key words for the talk: dualizing complexes, tilting complexes, Morita equivalence, rigid dualizing complexes, Gorenstein

Please summarize the lecture in 5 or fewer sentances: Discuss specialized topics such as: dualizing complexes (commutative and non-commutative), tilting complexes (two-sided), derived Morita theory (for DG algebras) and perserve sheave's, rigid dualizing complexes.

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(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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Take any $M \in \mathsf{D}(\mathsf{Mod}\, A)$. Because $A$ is commutative we have a triangulated functor

$$\mathrm{R}\,\mathrm{Hom}_A(-, M) : \mathsf{D}(\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathsf{D}(\mathsf{Mod}\, A).$$

Cf. Example 4.10.

**Definition 6.1.** A dualizing complex over $A$ is a complex $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ with finite injective dimension, such that the canonical morphism

$$A \to \mathrm{R}\,\mathrm{Hom}_A(R, R)$$

in $\mathsf{D}(\mathsf{Mod}\, A)$ is an isomorphism.

If we choose a bounded injective resolution $R \to I$, then there is an isomorphism of triangulated functors

$$\mathrm{R}\,\mathrm{Hom}_A(-, R) \cong \mathrm{Hom}_A(-, I).$$

**Example 6.2.** Assume $A$ is a Gorenstein ring, namely the free module $R := A$ has finite injective dimension.

There are plenty of Gorenstein rings; for instance any regular ring is Gorenstein.

Then $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$, and the reflexivity condition holds:

$$\mathrm{R}\,\mathrm{Hom}_A(R, R) \cong \mathrm{Hom}_A(R, R) \cong A.$$

We see that the module $R = A$ is a dualizing complex over the ring $A$.

Here are several important results from [Ha].

**Theorem 6.3. (Duality)** Suppose $R$ is a dualizing complex over $A$. Then the triangulated functor

$$\mathrm{R}\,\mathrm{Hom}_A(-, R) : \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$$

is an equivalence.

**Theorem 6.4. (Uniqueness)** Suppose $R$ and $R'$ are dualizing complexes over $A$, and $\mathrm{Spec}\, A$ is connected. Then there is an invertible module $P$ and an integer $n$ such that $R' \cong R \otimes_A P[n]$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

**Theorem 6.5. (Existence)** If $A$ has a dualizing complex, and $B$ is a finite type $A$-algebra, then $B$ has a dualizing complex.

## 7. Noncommutative Dualizing Complexes

In this section $A$ is a noncommutative noetherian ring. (This is short for: $A$ is not-necessarily-commutative, and left-and-right noetherian.)

For technical reasons we assume that $A$ is an algebra over a field $\mathbb{K}$.

We denote by $A^{\mathrm{op}}$ the opposite algebra (the same addition, but multiplication is reversed), and by $A^{\mathrm{e}} := A \otimes_{\mathbb{K}} A^{\mathrm{op}}$ the enveloping algebra.

Thus $\mathsf{Mod}\, A^{\mathrm{op}}$ is the category of right $A$-modules, and $\mathsf{Mod}\, A^{\mathrm{e}}$ is the category of $\mathbb{K}$-central $A$-bimodules.

Any $M \in \mathsf{Mod}\, A^{\mathrm{e}}$ gives rise to $\mathbb{K}$-linear functors

$$\mathrm{Hom}_A(-, M) : (\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathsf{Mod}\, A^{\mathrm{op}}$$

and

$$\mathrm{Hom}_{A^{\mathrm{op}}}(-, M) : (\mathsf{Mod}\, A^{\mathrm{op}})^{\mathrm{op}} \to \mathsf{Mod}\, A.$$

These functors can be derived, yielding $\mathbb{K}$-linear triangulated functors

$$\mathrm{R}\,\mathrm{Hom}_A(-, M) : \mathbf{D}(\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{op}})$$

and

$$\mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{op}}}(-, M) : \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{op}})^{\mathrm{op}} \to \mathbf{D}(\mathsf{Mod}\, A).$$

One way to construct these derived functors is to choose a quasi-isomorphism $M \to I$ in $\mathbf{K}(\mathsf{Mod}\, A^{\mathrm{e}})$, with $I$ a complex that is K-injective on both sides, i.e. over $A$ and over $A^{\mathrm{op}}$.

Then

$$\mathrm{R}\,\mathrm{Hom}_A(-, M) \cong \mathrm{Hom}_A(-, I)$$

and

$$\mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{op}}}(-, M) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(-, I).$$

The reason that we need $\mathbb{K}$ to be a field is to insure that such "bi-K-injective" resolutions $M \to I$ exist.



Note that even if $A$ is commutative, this setup is still meaningful – not all $A$-bimodules are $A$-central!

**Definition 7.1.** ([Ye1]) A noncommutative dualizing complex over $A$ is a complex $R \in \mathbf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$ satisfying these three conditions:

(i) The cohomology modules $\mathrm{H}^q(R)$ are finitely generated over $A$ and over $A^{\mathrm{op}}$.

(ii) The complex $R$ has finite injective dimension over $A$ and over $A^{\mathrm{op}}$.

(iii) The canonical morphisms

$$A \to \mathrm{R}\,\mathrm{Hom}_A(R, R)$$

and

$$A \to \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{op}}}(R, R)$$

in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ are isomorphisms.



Condition (ii) implies that $R$ has a "bounded bi-injective resolution", namely there is a quasi-isomorphism $R \to I$ in $\mathbf{K}(\mathsf{Mod}\, A^{\mathrm{e}})$, with $I$ a bounded complex of bimodules that are injective on both sides.

**Theorem 7.2.** (**Duality**, [Ye1]) Suppose $R$ is a noncommutative dualizing complex over $A$. Then the triangulated functor

$$\mathrm{R}\,\mathrm{Hom}_A(-, R) : \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{op}})$$

is an equivalence, with quasi-inverse $\mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{op}}}(-, R)$.

Existence and uniqueness are much more complicated than in the noncommutative case. I will talk about them later.



**Example 7.3.** The noncommutative ring $A$ is called Gorenstein if the bimodule $A$ has finite injective dimension on both sides.

It is not hard to see that $A$ is Gorenstein iff it has a noncommutative dualizing complex of the form $P[n]$, for some integer $n$ and invertible bimodule $P$.

Here invertible bimodule is in the sense of Morita theory, namely there is another bimodule $P^{\vee}$ such that

$$P \otimes_A P^{\vee} \cong P^{\vee} \otimes_A P \cong A$$

in $\mathsf{Mod}\, A^{\mathrm{e}}$.

Any regular ring is Gorenstein.

For more results about noncommutative Gorenstein rings see [Jo] and [JZ].

## 8. Tilting Complexes and Derived Morita Theory

Let $A$ and $B$ be noncommutative $\mathbb{K}$-algebras.

Suppose $M \in \mathbf{D}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$ and $N \in \mathbf{D}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$.

The left derived tensor product

$$M \otimes^{\mathrm{L}}_{B} N \in \mathbf{D}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} A^{\mathrm{op}})$$

exists.

It can be constructed by choosing a resolution $P \to M$ in $\mathbf{K}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$, where $P$ is a complex that's K-projective over $B^{\mathrm{op}}$;

or by choosing a resolution $Q \to N$ in $\mathbf{K}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$, where $Q$ is a complex that's K-projective over $B$.



Here is a definition generalizing the notion of invertible bimodule. It is due to Rickard [Ri1], [Ri2].

**Definition 8.1.** A complex

$$T \in \mathbf{D}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$$

is called a two-sided tilting complex over $A$-$B$

if there exists a complex

$$T^{\vee} \in \mathbf{D}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$$

such that

$$T \otimes^{\mathrm{L}}_{B} T^{\vee} \cong A$$

in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, and

$$T^{\vee} \otimes^{\mathrm{L}}_{A} T \cong B$$

in $\mathbf{D}(\mathsf{Mod}\, B^{\mathrm{e}})$.

When $B = A$ we say that $T$ is a two-sided tilting complex over $A$.



The complex $T^{\vee}$ is called the inverse of $T$. It is unique up to isomorphism in $\mathbf{D}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$. Indeed we have this result:

**Proposition 8.2.** Let $T$ be a two-sided tilting complex.

1. The inverse $T^{\vee}$ is isomorphic to $\mathrm{R}\,\mathrm{Hom}_{A}(T, A)$.
2. $T$ has a bounded bi-projective resolution $P \to T$.

**Definition 8.3.** The algebras $A$ and $B$ are said to be derived Morita equivalent if there is a $\mathbb{K}$-linear triangulated equivalence

$$\mathbf{D}(\mathsf{Mod}\, A) \approx \mathbf{D}(\mathsf{Mod}\, B).$$

**Theorem 8.4.** ([Ri2]) The $\mathbb{K}$-algebras $A$ and $B$ are derived Morita equivalent iff there exists a two-sided tilting complex over $A$-$B$.



Here is a result relating dualizing complexes and tilting complexes.

**Theorem 8.5.** (**Uniqueness**, [Ye3]) Suppose $R$ and $R'$ are noncommutative dualizing complexes over $A$.

Then the complex

$$T := \mathrm{R}\,\mathrm{Hom}_{A}(R, R')$$

is a two-sided tilting complex over $A$, and

$$R' \cong R \otimes^{\mathrm{L}}_{A} T$$

in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

It is easy to see that if $T_1$ and $T_2$ are two-sided tilting complexes over $A$, then so is $T_1 \otimes^{\mathrm{L}}_A T_2$.

This leads to the next definition.

**Definition 8.6.** ([Ye3]) Let $A$ be a noncommutative $\mathbb{K}$-algebra.

The derived Picard group of $A$ is the group $\mathrm{DPic}(A)$ whose elements are the isomorphism classes (in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$) of two-sided tilting complexes.

The multiplication is

$$[T_1] \cdot [T_2] := [T_1 \otimes^{\mathrm{L}}_A T_2],$$

and the inverse is

$$[T]^{-1} := \mathrm{R}\mathrm{Hom}_A(T, A).$$



Here is a consequence of Theorem 8.5.

**Corollary 8.7.** Suppose the noncommutative $\mathbb{K}$-algebra $A$ has at least one dualizing complex.

Then the right action

$$[R] \cdot [T] := [R \otimes^{\mathrm{L}}_A T]$$

of the group $\mathrm{DPic}(A)$ on the set of isomorphism classes of dualizing complexes is simply transitive.

It is natural to ask about the structure of the group $\mathrm{DPic}(A)$.



**Theorem 8.8.** ([RZ], [Ye3]) If the ring $A$ is either commutative (with connected spectrum) or local, then

$$\mathrm{DPic}(A) \cong \mathrm{Pic}(A) \times \mathbb{Z}.$$

Here $\mathrm{Pic}(A)$ is the noncommutative Picard group of $A$, made up of invertible bimodules.

For nonlocal noncommutative rings the group $\mathrm{DPic}(A)$ is bigger. See the paper [MY] for some calculations. These calculations are related to CY-dimensions of some rings; cf. Example 9.7.



## 9. Rigid Dualizing Complexes

The material in this final section is largely due to Van den Bergh [VdB1]. His results were extended by J. Zhang and myself.

Again $A$ is a noetherian noncommutative algebra over a field $\mathbb{K}$, and $A^{\mathrm{e}} = A \otimes_{\mathbb{K}} A^{\mathrm{op}}$.

Take $M \in \mathsf{Mod}\, A^{\mathrm{e}}$. Then the $\mathbb{K}$-module $M \otimes_{\mathbb{K}} M$ has four commuting actions by $A$, which we arrange as follows.

The algebra $A^{\mathrm{e};\mathrm{in}} := A^{\mathrm{e}}$ acts on $M \otimes_{\mathbb{K}} M$ by

$$(a_1 \otimes a_2) \cdot_{\mathrm{in}} (m_1 \otimes m_2) := (m_1 \cdot a_2) \otimes (a_1 \cdot m_2),$$

and the algebra $A^{\mathrm{e};\mathrm{out}} := A^{\mathrm{e}}$ acts by

$$(a_1 \otimes a_2) \cdot_{\mathrm{out}} (m_1 \otimes m_2) := (a_1 \cdot m_1) \otimes (m_2 \cdot a_2).$$

The bimodule $A$ is viewed as an object of $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ in the obvious way.

Now take $M \in \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$. We define the square of $M$ to be the complex

$$\mathrm{Sq}(M) := \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e};\,\mathrm{out}}}(A, M \otimes_{\mathbb{K}} M) \in \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e};\,\mathrm{in}}).$$

We get a functor

$$\mathrm{Sq} : \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}}) \to \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}}).$$

This is not an additive functor. Indeed, it is a quadratic functor: given an element $a \in \mathrm{Z}(A)$ and a morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, one has

$$\mathrm{Sq}(a\phi) = \mathrm{Sq}(\phi a) = a^2\, \mathrm{Sq}(\phi).$$

Let $(M, \rho)$ and $(N, \sigma)$ be rigid complexes over $A$.

A rigid morphism

$$\phi : (M, \rho) \to (N, \sigma)$$

is a morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \rho\ } & \mathrm{Sq}(M) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \mathrm{Sq}(\phi)} \\ N & \xrightarrow{\ \sigma\ } & \mathrm{Sq}(N) \end{array}$$

is commutative.

**Theorem 9.2.** (**Uniqueness**, [VdB1], [Ye3]) Suppose $(R, \rho)$ and $(R', \rho')$ are both rigid dualizing complexes over $A$. Then there is a unique rigid isomorphism

$$\phi : (R, \rho) \xrightarrow{\simeq} (R', \rho').$$

Note that the cohomologies of $\mathrm{Sq}(M)$ are

$$\mathrm{H}^q(\mathrm{Sq}(M)) = \mathrm{Ext}^q_{A^{\mathrm{e}}}(A, M \otimes_{\mathbb{K}} M),$$

so they are precisely the Hochschild cohomologies of $M \otimes_{\mathbb{K}} M$.

A rigid complex over $A$ (relative to $\mathbb{K}$) is a pair $(M, \rho)$ consisting of a complex $M \in \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, and an isomorphism

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}(M)$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

**Definition 9.1.** ([VdB1]) A rigid dualizing complex over $A$ (relative to $\mathbb{K}$) is a rigid complex $(R, \rho)$ such that $R$ is a dualizing complex.

As for existence, let me first give an easy case.

**Proposition 9.3.** If $A$ is finite over its center, and is finitely generated as $\mathbb{K}$-algebra, then $A$ has a rigid dualizing complex.

Actually, in this case it is quite easy to write down a formula for the rigid dualizing complex.

In the next existence result, by a filtration $F = \{F_i(A)\}_{i \in \mathbb{Z}}$ of the algebra $A$ we mean an ascending exhaustive nonnegative filtration.

Such a filtration gives rise to a graded $\mathbb{K}$-algebra

$$\mathrm{gr}^F(A) = \bigoplus_{i \geq 0} \mathrm{gr}^F_i(A).$$

**Theorem 9.4.** (**Existence**, [VdB1], [YZ3]) Suppose $A$ admits a filtration $F$ such that $\mathrm{gr}^F(A)$ is finite over its center and finitely generated as $\mathbb{K}$-algebra. Then $A$ has a rigid dualizing complex.

This theorem applies to the ring of differential operators $\mathcal{D}(C)$, where $C$ is a smooth commutative $\mathbb{K}$-algebra (and $\operatorname{char} \mathbb{K} = 0$).

It also applies to any quotient of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$.

I will finish with some examples.

**Example 9.5.** Let $A$ be a noetherian $\mathbb{K}$-algebra satisfying these two conditions:

- $A$ is smooth, namely the $A^{\mathrm{e}}$-module $A$ has finite projective dimension.
- There is an integer $n$ such that

$$\mathrm{Ext}^q_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \cong \begin{cases} A & \text{if } q = n \\ 0 & \text{otherwise.} \end{cases}$$

Then $A$ is a regular ring, and the complex $R := A[n]$ is a rigid dualizing complex over $A$.

Such an algebra $A$ is called an $n$-dimensional Artin-Schelter regular algebra, or an $n$-dimensional Calabi-Yau algebra.

**Example 9.6.** Let $\mathfrak{g}$ be an $n$-dimensional Lie algeba, and $A := \mathrm{U}(\mathfrak{g})$, the universal enveloping algebra.

Then the rigid dualizing complex of $A$ is $R := A^{\sigma}[n]$, where $A^{\sigma}$ is the trivial bimodule $A$, twisted on the right by an automorphism $\sigma$.

Using the Hopf structure of $A$ we can express $A^{\sigma}$ like this:

$$A^{\sigma} \cong \mathrm{U}(\mathfrak{g}) \otimes_{\mathbb{K}} \textstyle\bigwedge^n \mathfrak{g},$$

the twist by the 1-dimensional representation $\bigwedge^n \mathfrak{g}$.

So $A$ is a twisted Calabi-Yau algebra.

If $\mathfrak{g}$ is semi-simple then there is no twist, so $A$ is Calabi-Yau. This was used by Ven den Bergh in his duality for Hochschild (co)homology [VdB2].

**Example 9.7.** Let

$$A := \begin{bmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{bmatrix}$$

the $2 \times 2$ matrix algebra.

The rigid dualizing complex here is

$$R := \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K}).$$

It is known that

$$R \otimes^{\mathrm{L}}_A R \otimes^{\mathrm{L}}_A R \cong A[1]$$

in $\mathbf{D}(\mathrm{Mod}\, A^{\mathrm{e}})$.

So $A$ is a Calabi-Yau algebra of dimension $\frac{1}{3}$.

**- END -**

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