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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Paul Smith

Talk Title: Introduction to non-commutative algebraic geometry II

Date: 01 / 29 / 13 **Time:** 2 : 00 am / [✓ pm] (circle one)

List 6-12 key words for the talk: Zhang twist, ample pair, homogeneous coordinate rings, stacks, AS-regular algebras, Sklyanin algebras

Please summarize the lecture in 5 or fewer sentances: Continue introducing non-commutative (NC) analogues of objects and compare and contrast between commutative and NC case. Discuss Artin-Zhang theorem, Rogalski-Zhang theorem, Artin-Stafford and Reiten-Van den Bergh theorems on projective NC curves, NC geometry and stacks, classification of AS-regular algebras, and intersection theory on NC projective surfaces.

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Noncommutative Algebraic Geometry
and
Algebraic Noncommutative Geometry
Second Lecture

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Introductory Workshop:
Noncommutative Algebraic Geometry and Representation Theory

MSRI

Reminder I

Theorem (Serre, 1955, FAC, §59)

If A is a commutative k-algebra generated by a finite number of elements of degree 1, then

$$\mathfrak{qgr}(S/I) \equiv \mathfrak{coh}\big(\mathrm{Proj}(A)\big),$$

the category of coherent sheaves on $\mathrm{Proj}(A)$*.*

If A is noetherian, more generally graded-coherent, not-necessarily commutative, we define

$$\mathfrak{qgr}(A) := \frac{\mathfrak{gr}(A)}{\mathfrak{fdim}(A)}$$

and define $\mathrm{Proj}_{nc}(A)$ implicitly by declaring

$$\mathfrak{coh}\big(\mathrm{Proj}_{nc}(A)\big) := \mathfrak{qgr}(A).$$

Reminder II. An algebraic variety can have a nc hcr

- Verevkin's Theorem $\Longrightarrow \mathfrak{Qcoh}(\mathbb{P}^1 \times \mathbb{P}^1) \equiv \mathfrak{QGr}(B)$ where

$$B := \frac{k\langle x,y\rangle}{[x^2,y]=[y^2,x]=0} = \frac{k\langle x,y\rangle}{(x^2y-yx^2, xy^2-y^2x)}.$$

- $\mathfrak{Qcoh}(\mathbb{P}^1) \equiv \mathfrak{QGr}(A)$ when

$$A = \frac{k\langle x,y\rangle}{(yx-xy-x^2)} \quad OR \quad \frac{k\langle x,y\rangle}{(yx-qxy)} \quad \text{for some } q \in k^\times$$

Reason: A is a Zhang twist of $k[X,Y]$, i.e., $A \cong k[X,Y]^\sigma$ for a suitable $\sigma \in \mathrm{GL}(kX+kY)$

Zhang twists

A = a graded ring

$\sigma : A \longrightarrow A$ is an algebra automorphism s.t. $\sigma(A_n) = A_n$ for all n.

$A^\sigma := A$ with new multiplication $a * b := a\sigma^m(b)$ if $a \in A_m$

Theorem (Zhang, 1992)

1. $\mathfrak{Gr}(A^\sigma) \equiv \mathfrak{Gr}(A)$ *via* $M \mapsto M^\sigma = M$ *with new* A*-action,* $m * a := m\sigma^i(a)$ *if* $m \in M_i$
2. $\mathfrak{QGr}(A^\sigma) \equiv \mathfrak{QGr}(A)$
3. $\operatorname{Proj}_{nc}(A^\sigma) \cong \operatorname{Proj}_{nc}(A)$

$$\text{Serre + Zhang} \Longrightarrow \mathfrak{Qcoh}(\mathbb{P}^1) \equiv \mathfrak{QGr}\big(k[X,Y]\big) \equiv \mathfrak{QGr}\big(k[X,Y]^\sigma\big).$$

Looking ahead: (2) is a special case of a deeper result due to Artin-Zhang.

$A^\sigma \cong B(\mathfrak{Gr}(A), A_A, s)$ for a suitable auto-equivalence $s : \mathfrak{Gr}(A) \to \mathfrak{Gr}(A)$.

e.g., a nc hcr for the degree 6 del Pezzo surface

$X := \mathbb{P}^2$ blown up at 3 non-colinear points
$= $ the degree 6 del Pezzo surface

If $\quad B = \dfrac{k\langle x,y\rangle}{(x^5 = yxy, y^2 = xyx)}$ with $\deg(x,y) = (1,2)$, then

$$\mathfrak{QGr}(B) \equiv \mathfrak{Qcoh}(X)$$

- $B = B(X, \mathcal{L}, \sigma)$ is a twisted hcr of X (to be defined soon)
- $\mathcal{L} = \mathcal{O}_X(-E)$ where
- E is a (-1)-curve, and
- $\sigma \in \mathrm{Aut}(X)$ has order 6 and cyclicly permutes the (-1)-curves
- A is 3-dim'l AS-regular & $H(A;t) = (1-t)^{-1}(1-t^2)^{-1}(1-t^3)^{-1}$.

The construction $B(\mathfrak{C}, \mathcal{O}, s)$

- $\mathfrak{C}$ = k-linear abelian category
- $\mathcal{O} \in \mathrm{Ob}(\mathfrak{C})$
- $s : \mathfrak{C} \to \mathfrak{C}$ is an autoequivalence/automorphism
- $\mathcal{O}(n) := s^n\mathcal{O}$

$$\text{Make} \qquad B(\mathfrak{C}, \mathcal{O}, s) := \bigoplus_{n=0}^{\infty} \mathrm{Hom}_{\mathfrak{C}}(\mathcal{O}, \mathcal{O}(n)) \quad \text{a graded ring}$$

with mult. $B_m \times B_n \longrightarrow B_{m+n}$ given by $(a, b) \mapsto ab := s^n(a) \circ b$.

$$\begin{tikzcd} \mathcal{O} \arrow[r, "b"] & \mathcal{O}(n) \arrow[r, "s^n(a)"] & \mathcal{O}(m+n) \end{tikzcd}$$

The Artin-Zhang Theorem I

Suppose $\mathfrak{C}$ is locally noetherian and $\mathcal{O}$ is noetherian.

The pair $(\mathcal{O}, s)$ is ample if

- for every noetherian $\mathcal{M} \in \operatorname{Ob}(\mathfrak{C})$ there are integers $r_1, \ldots, r_n$ and an epimorphism $\oplus_{i=1}^{n} \mathcal{O}(-r_i) \twoheadrightarrow \mathcal{M}$ and
- for every epimorphism $\mathcal{M} \to \mathcal{N}$ between noetherian objects, the induced map $\operatorname{Hom}(\mathcal{O}(-n), \mathcal{M}) \to \operatorname{Hom}(\mathcal{O}(-n), \mathcal{N})$ is surjective for $n \gg 0$.

Theorem (Artin-Zhang)

Let $(\mathcal{O}, s)$ be an ample pair in $\mathfrak{C}$ and suppose $\dim_k \operatorname{Hom}_{\mathfrak{C}}(\mathcal{M}, \mathcal{N}) < \infty$ for all noetherian $\mathcal{M}$ and $\mathcal{N}$. Then

- $\mathfrak{C} \equiv \mathfrak{QGr}(B(\mathfrak{C}, \mathcal{O}, s))$ *via* $\mathcal{M} \mapsto \oplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathfrak{C}}(\mathcal{O}, s^n \mathcal{M})$
- $B(\mathfrak{C}, \mathcal{O}, s)$ *is right noetherian*
- $B(\mathfrak{C}, \mathcal{O}, s)$ *satisfies the χ_1 condition*

The Artin-Zhang Theorem II: a converse

Theorem (Artin-Zhang)

Let B be a finitely graded, right noetherian k-algebra that satisfies χ_1. Let $(\mathfrak{C}, \mathcal{O}, s) = (\mathfrak{QGr}(B), \pi^* B, (1))$. Then

- $(\pi^* B, (1))$ *is ample*
- *dim*$_k$ $\operatorname{Hom}_{\mathfrak{C}}(M, N) < \infty$ *for all* $M, N \in \operatorname{gr}(B)$
- *there is a natural homomorphism* $B \to B(\mathfrak{C}, \mathcal{O}, s)$ *with finite dimensional kernel and cokernel*
- $\mathfrak{QGr}(B) \equiv \mathfrak{QGr}\big(B(\mathfrak{C}, \mathcal{O}, s)\big)$.

Theorem (Polishchuk)

Vague statement: there is a generalization of the Artin-Zhang theorems for coherent graded algebras where the ample pair $(\mathcal{O}, s)$ is replaced by an ample sequence $\{\mathcal{E}_n \mid n \in \mathbb{Z}\}$.

Homogeneous coordinate rings (hcrs) in algebraic geometry

$X =$ quasi-projective variety/scheme
$\mathcal{L} =$ an invertible $\mathcal{O}_X$-module
$\sigma \in \operatorname{Aut}(X)$
$\mathcal{L}$ is ample if the natural map $\mathcal{O}_X \otimes H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is an epimorphism for all $n \gg 0$.

Theorem (Serre)

If $\mathcal{L}$ is ample, then

1. $B = B(X, \mathcal{L}, \mathrm{id}) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$ *is a fin. gend. k-algebra, so noetherian;*
2. $\mathfrak{QGr}(B) \equiv \mathfrak{Qcoh}(X)$*;*
3. $X \cong \operatorname{Proj}(B)$.

Twisted hcrs in nc algebraic geometry

$\mathcal{L}_\sigma^n := \mathcal{L} \otimes \sigma^*\mathcal{L} \otimes \cdots \otimes (\sigma^*)^{n-1}\mathcal{L}$
$\mathcal{L}$ is σ-ample if the natural map $\mathcal{O}_X \otimes H^0(X, \mathcal{F} \otimes \mathcal{L}_\sigma^{\otimes n}) \to \mathcal{F} \otimes \mathcal{L}_\sigma^{\otimes n}$ is an epimorphism for all $n \gg 0$.

$$B(X, \mathcal{L}, \sigma) = \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}_\sigma^n)$$

with mult. $f \bullet g = f \cdot (\sigma^m)^*(g)$ when $f \in B_m$ is a twisted hcr

Theorem (Artin-Van den Bergh)

If $\mathcal{L}$ is σ-ample, then

1. $B = B(X, \mathcal{L}, \sigma)$ *is a fin. gend. noetherian k-algebra;*
2. $\mathfrak{QGr}(B) \equiv \mathfrak{Qcoh}(X)$;
3. X *integral* $\Longrightarrow$ $\operatorname{Fract}_{\mathrm{gr}}\left(B(X, \mathcal{L}, \sigma)\right) \cong k(X)[t, t^{-1}; \sigma]$.

The point scheme

Let A be a locally finite connected graded k-algebra.

R a commutative k-algebra

An R–point module for A is a cyclic graded right $A \otimes_k R$-module

$$M = M_0 \oplus M_1 \oplus \cdots$$

s.t. $M_0 = R$ and M_n is a locally free R-module of rank 1 for all $n \geq 1$.

Define $F : \mathfrak{Cat}(\text{commutative } k\text{-algebras}) \longrightarrow \mathfrak{Sets}$ by

$$F(R) := \{\text{isoclasses of } R\text{–point modules for } A\}$$

A quasi-projective scheme X representing F, if it exists, is called the point scheme for A.

Theorem (Artin-Zhang)

The point scheme for A exists if A is strongly noetherian.

Ubiquity of $B(X, \mathcal{L}, \sigma)$

Theorem (Rogalski-Zhang)

Suppose k is algebraically closed and A a strongly noetherian connected graded k-algebra s.t. $A = k[A_1]$. Let X = the point scheme for A.

1. *There exists $\sigma \in \operatorname{Aut}(X)$ and an invertible σ-ample $\mathcal{O}_X$-module $\mathcal{L}$ and a canonical graded ring homomorphism $\Phi : A \to B(X, \mathcal{L}, \sigma)$.*
2. *Φ is surjective in large degree.*
3. *In large degree, $\ker(\Phi) =$*
   $\{a \in A \mid Ma = 0\ \forall$ comm. k-algs R and all R-point modules $M\}$.
4. *$\{R$-point modules for $A\} = \{R$-point modules for $B\}$*
5. *if S is a commutative ring, $\tau \in \operatorname{Aut}(A)$, then every graded k-algebra homom. $\varphi : A \to S[t; \tau]$ factors through Φ (up to finite dimension).*

The strongly noetherian property

A is strongly left noetherian if $A \otimes_k R$ is left noetherian for all noetherian commutative k-algebras R.

Resco-Small: $\exists$ a noetherian finitely generated k-algebra that is not strongly noetherian; their example is not graded and $k \neq \overline{k}$

Theorem (Rogalski)

Let S be a generic Zhang twist of a polynomial ring in ≥ 3 variables over $k = \overline{k}$. Let $V \subset S_1$ be a generic codimension-one subspace. Then $k[V] \subset S$ is not strongly noetherian.

The generic point of $\mathrm{Proj}_{nc}(A)$ when A is a domain

Let A be a graded k-algebra that is a domain. If A does not contain a graded subalgebra isomorphic to the free algebra on two variables (e.g., if A is right noetherian) it has a graded ring of fractions,

$$\mathrm{Fract}_{\mathrm{gr}}(A) := \{ab^{-1} \mid a, b \in A \text{ and } b \text{ is non-zero and homogeneous}\}.$$

Its degree-zero subalgebra,

$$D := \{ab^{-1} \mid \exists\, n \text{ such that } a, b \in A_n \text{ and } b \neq 0\},$$

is a division ring and, if $t \in A_1 - \{0\}$, $\mathrm{Fract}_{\mathrm{gr}}(A) \cong D[t, t^{-1}; \sigma]$ where $\sigma \in \mathrm{Aut}(D)$.

D is determined by $\mathfrak{QGr}(A)$ and if D is commutative σ depends only on the degree shift (1).

Vague: the dynamical properties of σ have a strong influence on the properties of A and $\mathrm{Proj}_{nc}(A)$.

Projective nc curves I

If X is an irreducible curve, $\mathcal{L}$ is σ-ample $\Longleftrightarrow$ it is ample.

Theorem (Artin-Stafford)

Let A be a graded domain s.t. $A = k[A_1]$ and $\operatorname{GKdim}_k(A_n) = 2$, then there is an irreducible curve Y, an invertible $\mathcal{O}_Y$-module $\mathcal{L}$, and $\sigma \in \operatorname{Aut}(Y)$, s.t.

1. $\Phi : A \hookrightarrow B(Y, \mathcal{L}, \sigma)$ *and* $\dim_k(\operatorname{coker}(\Phi)) < \infty$
2. $\operatorname{Fract}_{\mathrm{gr}}(A) \cong k(Y)[t, t^{-1}; \sigma]$
3. *A is noetherian*
4. *A is finite over its center* $\Longleftrightarrow$ $|\sigma| < \infty$
5. $\mathfrak{QGr}(A) \equiv \mathfrak{Qcoh}(Y)$

Roughly: if X_{nc} is a nc curve, there is a commutative curve Y and a birational isomorphism $X_{nc} \twoheadrightarrow Y$

cf. DM stack $\mathcal{X} \twoheadrightarrow X =$ the coarse moduli space of $\mathcal{X}$

An abstract definition of projective nc curves

If X is a smooth, proper and connected k-scheme of dimension n, then

1. $\mathfrak{coh}(X)$ is noetherian
2. $\mathfrak{coh}(X)$ is Ext-finite, i.e., $\dim_k \operatorname{Ext}^q(\mathcal{M},\mathcal{N}) < \infty$ for all $\mathcal{M},\mathcal{N} \in \operatorname{Ob}(\mathfrak{coh}(X))$ and all q;
3. n is minimal such that $\operatorname{Ext}^q(\mathcal{M},\mathcal{N}) = 0$ for all $\mathcal{M},\mathcal{N} \in \operatorname{Ob}(\mathfrak{coh}(X))$ and all $q > n$;
4. $\mathfrak{coh}(X)$ satisfies Serre duality, i.e, there is an autoequivalence $\mathbb{S} : \mathrm{D}^b(\mathfrak{coh}(X)) \longrightarrow \mathrm{D}^b(\mathfrak{coh}(X))$ s.t. there are isomorphisms $\operatorname{Hom}(\mathcal{M},\mathcal{N}) \cong \operatorname{Hom}(\mathcal{N},\mathbb{S}\mathcal{M})^*$, natural in $\mathcal{M},\mathcal{N} \in \mathrm{D}^b(\mathfrak{coh}(X))$;
5. $\mathfrak{coh}(X)$ is saturated, i.e., every cohomological functor $H : \mathrm{D}^b(\mathfrak{coh}(X)) \to \operatorname{mod}(k)$ of finite type is of the form $\operatorname{Hom}(\mathcal{M},-)$.

H has finite type if $\{n \in \mathbb{Z} \mid H(\mathcal{F}[n]) \neq 0\}$ is finite for all $\mathcal{F} \in \mathrm{D}^b(\mathfrak{coh}(X))$

An abelian category $\mathfrak{C}$ is connected if $\mathfrak{C} \neq \mathfrak{C}_1 \oplus \mathfrak{C}_2$ in a non-trivial way

nc projective curves up to derived equivalence

Theorem (Reiten-Van den Bergh)

If $\mathfrak{C}$ is a saturated, connected, noetherian, Ext-finite, hereditary category satisfying Serre duality over k, then $\mathfrak{C}$ is either

1. $\mathfrak{mod}(\Lambda)$ *where Λ is an indecomposable finite dim'l hereditary algebra or*
2. $\mathfrak{coh}(\mathcal{A})$ *where $\mathcal{A}$ is a sheaf of hereditary $\mathcal{O}_X$ orders over a smooth connected projective curve X.*

Furthermore, the categories in (1) and (2)

3. *have the abstract properties in the hypothesis of the previous sentence and*
4. *are $\equiv \mathfrak{qgr}(A)$ for some graded k-algebra A s.t. $\operatorname{GKdim}(A) \leq 2$.*

We can interpret this result as a description of all smooth connected projective curves up to derived equivalence.

The nc curves in (2) are stacks.

NCG and stacks I

Stacks turn up often in nc (algebraic) geometry, e.g., the exceptional fiber of Van den Bergh's blowup of a suitable point on a suitable nc surface is a stack (previous talk) and the nc curves in part (2) of Reiten and Van den Bergh's classification are stacks.

Operator algebras: Given a groupoid $s,t : Y \rightrightarrows X$ in $\mathfrak{Top}$, there is an associated (typically non-commutative) C*-algebra that behaves as if it is the ring of functions on the associated quotient. Many important examples can be interpreted as arising in this way, e.g., graph C*-algebras—Cuntz-Krieger algebras are graph C*-algebras.

Algebraic geometry: Given a groupoid $s,t : Y \rightrightarrows X$ in $\mathfrak{Schemes}$, the associated quotient object $[X/Y]$ is called an algebraic stack. Stacks are

indispensable in the study of moduli problem, cf. the problem of non-closed orbits $X \longrightarrow [X/G] \twoheadrightarrow X/G$.

NCG and stacks II

Theorem (D.Chan-Ingalls)

Let $s, t : Y \rightrightarrows X$ *be a finite flat groupoid scheme with* X *a quasi-projective variety and coarse moduli space* S. *There is an associated sheaf,* $\mathcal{O}_{X/Y}$, *of (typically non-commutative)* $\mathcal{O}_S$*-algebras. When* X *is a smooth curve and the groupoid action is generically free* $\mathcal{O}_{X/Y}$ *is a hereditary order on* X. *Up to Morita equivalence all hereditary orders on smooth curves arise in this way.*

The result of Chan and Ingalls complements the Reiten-Van den Bergh classification: it establishes a natural correspondence between smooth proper 1-dimensional Deligne-Mumford stacks of finite type over k that are generically schemes and smooth non-commutative curves that do not have a progenerator. The correspondence is given by taking the category of quasi-coherent sheaves on the stack.

Weighted projective spaces, $\mathfrak{QGr}(-)$, and stacks

Serre's Theorem (previous talk) assumed $A = k[A_1]$ so does not apply to the polynomial ring $A := k[x_0, \ldots, x_n]$ with $\deg(x_i) = q_i \geq 1$.

Weighted projective space is $\mathbb{P}^n_{q_0,\ldots,q_n} := \operatorname{Proj}(A) \cong \mathbb{P}^n/G$ where $G = \mu_{q_0} \times \cdots \times \mu_{q_n}$ is a product of roots of unity with action

$$(\xi_0, \ldots, \xi_n) \cdot (a_0, \ldots, a_n) = (\xi_0 a_0, \ldots, \xi_n a_n).$$

$\mathbb{P}^n_{q_0,\ldots,q_n}$ is often singular e.g., $\mathbb{P}^2_{1,1,2}$.

Usually, $\mathfrak{QGr}(A) \not\equiv \mathfrak{Qcoh}(\mathbb{P}^n_{q_0,\ldots,q_n})$.

But $\mathfrak{QGr}(A)$ is better than $\mathfrak{Qcoh}(\mathbb{P}^n_{q_0,\ldots,q_n})$,

e.g., $\operatorname{Ext}^n_{\mathfrak{QGr}(A)}(-,-) = 0$ and $\mathfrak{QGr}(A)$ satisfies Serre duality.

The quotient morphism factors as $\mathbb{P}^n \to [\mathbb{P}^n/G] \to \mathbb{P}^n/G$ where $[\mathbb{P}^n/G]$ is the stack-theoretic quotient and

$$\mathfrak{Qcoh}([\mathbb{P}^n/G]) \equiv \mathfrak{QGr}(A).$$

General Principle: good homological properties of $A \Longrightarrow$ good homological properties of $\mathfrak{QGr}(A)$.

The graded Weyl algebra (Sierra)

Another example of a nice nc hcr for a stack.

e.g., if $D = \dfrac{\mathbb{C}\langle x, y\rangle}{(xy - yx - 1)} \cong \mathcal{D}(\mathbb{A}^1)$ with $\deg(x, y) = (1, -1)$, then

$$\mathfrak{QGr}(D) \equiv \mathfrak{Gr}(D) \equiv \mathfrak{Gr}(C, \mathbb{Z}_{\mathrm{fin}}) \equiv \mathfrak{Qcoh}\left[\frac{\operatorname{Spec} C}{G}\right],$$

the stack-theoretic quotient, where

- $C := \mathbb{C}[z]\left[\sqrt{z-n} \mid n \in \mathbb{Z}\right]$ is commutative
- $\mathbb{Z}_{\mathrm{fin}} := \{\text{finite subsets of } \mathbb{Z}\}$ with group operation exclusive or
- $\mathfrak{Gr}(C, \mathbb{Z}_{\mathrm{fin}}) := \mathbb{Z}_{\mathrm{fin}}$-graded C-modules
- $\deg(\sqrt{z-n}) := \{n\}$, and $G :=$ affine gp scheme $\operatorname{Spec}\mathbb{C}[\mathbb{Z}_{\mathrm{fin}}]$ where
- $\mathbb{C}[\mathbb{Z}_{\mathrm{fin}}] :=$ the group algebra with its standard Hopf algebra structure.
- $\{$isoclasses of simple graded D-modules$\}$ are parametrized by

  $$\cdots \text{———} : \text{———} : \text{———} : \text{———} : \text{———} : \text{———} \cdots$$

  $\mathbb{A}^1_{\mathbb{C}}$ with the stack $B\mathbb{Z}_2$ at each point in $\mathbb{Z} \subset \mathbb{A}^1_{\mathbb{C}}$.

AS-regular algebras

Artin and Schelter's philosophy: noncommutative analogues of $\mathbb{P}^2$ should be of the form $\text{Proj}_{nc}(A)$ where $A = k[A_1]$ is a graded k-algebra that is "like" the polynomial ring in three variables.

Definition [Artin-Schelter, 1987] A connected graded k-algebra, $A = k \oplus A_1 \oplus \cdots$, is Artin-Schelter regular of dimension d if

- $\text{gldim}(A) = d < \infty$,
- $\text{GKdim}(A) < \infty$, and
- $\text{Ext}^i_A(k_A, A) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

General Principle: good homological properties of $A \Longrightarrow$ good homological properties of $\mathfrak{QGr}(A)$ even if A is not commutative.

Classification of 3-dim'l AS-regular algebras s.t. $A = k[A_1]$

If A is 3-dim'l AS-regular algebra s.t. $A = k[A_1]$, then either

- $A = k\langle x, y\rangle/(2 \text{ cubic relations})$ and

  the min'l proj. res. of $k = A/A_{\geq 1}$ is

  $0 \to A(-4) \to A(-3)^2 \to A(-1)^2 \to A \to k \to 0$ OR
- $A = k\langle x, y, z\rangle/(3 \text{ quadratic relations})$ and

  the min'l proj. res. of $k = A/A_{\geq 1}$ is

  $0 \to A(-3) \to A(-2)^3 \to A(-1)^3 \to A \to k \to 0$

$\dim_k(A_1) = 2 \Longrightarrow H(A;t) = (1-t)^{-2}(1-t^2)^{-1}$
$\dim_k(A_1) = 3 \Longrightarrow H(A;t) = (1-t)^{-3}$
$\therefore H(A;t) = H(\text{polyn. rings on gen'ors of weights (1,1,2) or (1,1,1)})$

Theorem (Artin-Schelter)

If $k = \overline{k}$ and $A = k[A_1]$ there are 13 irreducible families of of 3-dim'l AS-regular algebras. Very explicit classification.

BUT are these rings noetherian?

Properties of 3-dim'l AS-regular algebras s.t. $A = k[A_1]$

Theorem (Artin-Tate-Van den Bergh, Stephenson)

Every 3-dim'l AS-regular algebra is a noetherian domain.

Theorem (Artin-Tate-Van den Bergh)

Suppose $A = k[A_1]$ *is 3-dim'l AS-regular with point scheme* E*. If* $\dim(E) = 2$*, then*

$$\mathfrak{QGr}(A) \equiv \begin{cases} \mathfrak{Qcoh}(\mathbb{P}^2) & \textit{if } \dim(A_1) = 3 \\ \mathfrak{Qcoh}(\mathbb{P}^1 \times \mathbb{P}^1) & \textit{if } \dim(A_1) = 2 \end{cases}$$

Theorem (Artin-Tate-Van den Bergh)

Suppose $A = k[A_1]$ is 3-dim'l AS-regular with point scheme E. If $\dim(E) \neq 2$, *then*

1. $\dim_k(A_1) = 2 \Longrightarrow E \overset{i}{\hookrightarrow} \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*) \cong \mathbb{P}^1 \times \mathbb{P}^1$
2. $\dim_k(A_1) = 3 \Longrightarrow E \overset{i}{\hookrightarrow} \mathbb{P}(A_1^*) \cong \mathbb{P}^2$
3. *$i(E)$ is an anti-canonical divisor*
4. *the function $M \rightsquigarrow M(1)_{\geq 0}$ on point modules induces $\sigma \in \mathrm{Aut}(E)$*

There is a surjective homom $\Phi : A \twoheadrightarrow B(E, \mathcal{L}, \sigma)$ where $\mathcal{L} = i^*\mathcal{O}(1,1)$ or $\mathcal{L} = i^*\mathcal{O}(1)$, and Φ induces a closed immersion $E \hookrightarrow \mathrm{Proj}_{nc}(A)$ embedding E as an effective divisor.

Non-commutative analogues of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$

Let $A = k[A_1]$ be a 3-dim'l AS-regular algebra. Philosophy:

- If $\dim_k(A_1) = 2$, $\mathrm{Proj}_{nc}(A)$ is a nc analogue of $\mathbb{P}^1 \times \mathbb{P}^1$.
- If $\dim_k(A_1) = 3$, $\mathrm{Proj}_{nc}(A)$ is a nc analogue of $\mathbb{P}^2$.

Van den Bergh replaced the vague phrase "nc analogue" by a precise notion of deformation:

Theorem (Van den Bergh)

If $A = k[A_1]$ is 3-dim'l AS-regular and $\dim_k(A_1) = 3$, then $\mathfrak{QGr}(A)$ is a deformation of $\mathfrak{Qcoh}(\mathbb{P}^2)$ and every deformation of $\mathfrak{Qcoh}(\mathbb{P}^2)$ is of this form *provided* $\mathrm{char}(k) \neq 3$.

This theorem vindicates Artin-Schelter's philosophy that nc analogues of $\mathbb{P}^2$ should have hcrs that are nc analogues of the polyn. ring on 3 variables.

Van den Bergh has classified nc deformations of $\mathbb{P}^1 \times \mathbb{P}^1$. The 3 dim'l AS-regular algebras with $\dim(A_1) = 2$ give some of them (a 2-dim'l family) one needs 3-dim'l AS-regular $\mathbb{Z}$-algebras as "homogeneous coordinate rings" to get all the deformations (a 3'diml family).

Geometry: points and lines on $\mathbb{P}^2_{\mathrm{nc}}$

Let $A = k[A_1]$ be a 3-dim'l AS-regular algebra such that $\dim_k(A_1) = 3$. A line module for A is an A-module L such that $L = L_0 A$ and $H(L;t) = (1-t)^{-2}$.
If L is a line module for A we call $\pi^* L$ a line on $\mathrm{Proj}_{nc}(A)$.

A line $\mathcal{L}$ passes through a point $p \in E$ if there is an epimorphism $\mathcal{L} \twoheadrightarrow \mathcal{O}_p$.

Theorem (Artin-Tate-Van den Bergh)

Let $A = k[A_1]$ be a 3-dim'l AS-regular algebra such that $\dim_k(A_1) = 3$. Let $E \subset \mathrm{Proj}_{nc}(A)$ be the point scheme.

1. *The lines in $\mathrm{Proj}_{nc}(A)$ are naturally parametrized by $\mathbb{P}(A_1)$.*
2. *If $p \in E$, there is a pencil of lines in $\mathrm{Proj}_{nc}(A)$ passing through p.*
3. *If $p, q \in E$ are distinct, there is a unique line in $\mathrm{Proj}_{nc}(A)$ passing through p and q.*

Up to isomorphism, $\{\text{line modules for } A\} = \{A/aA \mid a \in A_1 - \{0\}\}$

Geometry: points and lines on $(\mathbb{P}^1 \times \mathbb{P}^1)_{\mathrm{nc}}$

Let $A = k[A_1]$ be a 3-dim'l AS-regular algebra such that $\dim_k(A_1) = 2$. A line module for A is either

(1) an A-module L s.t. $L = L_0 A$ and $H(L;t) = (1-t)^{-1}(1-t^2)^{-1}$ or

(2) $L(-1)$ where L satisfies (1).

Lines coming from (1) (resp., (2)) are said to belong to the first (resp., second) ruling.

Theorem (Artin-Tate-Van den Bergh)

Let $A = k[A_1]$ be a 3-dim'l AS-regular algebra such that $\dim_k(A_1) = 2$. Let $E \subset \mathrm{Proj}_{nc}(A)$ be the point scheme.

1. *The lines in $\mathrm{Proj}_{nc}(A)$ belonging to each ruling are naturally parametrized by $\mathbb{P}(A_1^*)$.*
2. *If $p \in E$, there is a unique line in each ruling passing through p.*

Intersection theory on nc projective surfaces I

Theorem (Mori)

Let A be a connected graded noetherian k-algebra with $\operatorname{gldim}(A) < \infty$. *Then*

$$K_0\big(\operatorname{Proj}_{nc}(A)\big) \overset{\theta}{\cong} \frac{\mathbb{Z}[t^{\pm 1}]}{q(t)}$$

where $q(t) = H(A;t)^{-1}$, $\theta(\pi^* M) = H(M;t)H(A;t)^{-1}$ *for* $M \in \mathfrak{gr}(A)$. *For example, if* $\mathcal{O} = \pi^* A$, *then* $\theta[\mathcal{O}(n)] = t^{-n}$.

Theorem (Mori, Bézout's Theorem for $\mathbb{P}^2_{\mathrm{nc}}$)

Vague: Let A be a 3-dim'l AS-regular algebra s.t. $\dim_k(A_1) = 3$ *and write* $\mathbb{P}^2_{\mathrm{nc}}$ *for* $\operatorname{Proj}_{nc}(A)$.

1. *Define* $[M]\cdot[N] := -\sum_{q=0}^{2} \dim_k \operatorname{Ext}^q_{\mathfrak{QGr}(A)}(M,N)$ *on* $K_0(\mathbb{P}^2_{\mathrm{nc}})$
2. *If M and N are curve-modules in* $\operatorname{Pic}(\mathbb{P}^2_{\mathrm{nc}}) := F^1K_0/F^2K_0 \cong \mathbb{Z}$ *and* $\operatorname{Hom}_{\mathfrak{QGr}(A)}(M,N) = 0$ *(no common component), then* $[M]\cdot[N] = \deg(M)\deg(N)$.

Intersection theory on nc projective surfaces II

Results of Jorgensen that specialize to the classical commutative results.

- Let Y be an effective divisor, in the sense of Van den Bergh, on a "suitable" nc-surface $(X, \mathcal{O}_X)$.
- The first Chern class of Y is the operator $c(Y) : K_0(X) \to K_0(X)$, $[M] \mapsto [M] - [M(-Y)]$. Roughly, $c(Y) =$ "intersect with Y".
- If C is a curve-module on X, define the intersection multiplicity $\langle Y, [C] \rangle := \chi(c(Y)([C]))$, where $\chi =$Euler characteristic.
- Riemann-Roch:
  $\chi(\mathcal{O}_X(Y)) = \chi(\mathcal{O}_X) + \frac{1}{2}(\langle Y, [\mathcal{O}_Y] \rangle - \langle Y, [\omega] - [\mathcal{O}_X] \rangle)$.
- Genus formula: $2g(Y) - 2 = \langle Y, [\mathcal{O}_Y] \rangle + \langle Y, [\omega] - [\mathcal{O}_X] \rangle$ where $g(Y)$ is the genus of the nc-curve Y.
- Self-intersection formula: $\langle Y, [\mathcal{O}_Y] \rangle = \deg(\mathcal{N})$ where $\mathcal{N}$ is the normal bundle of Y in X.
- If $\alpha : \widetilde{X} \to X$ is Van den Bergh's blowup at a point $p \in X$ s.t. $E := \alpha^{-1}(p) \cong \mathbb{P}^1$, then $\langle E, [\mathcal{O}_E] \rangle = -1$ on $\widetilde{X}$.

Sklyanin algebras

3-dim'l Sklyanin algebras: A family of algebras $A_{a,b,c}$ parametrized by $(a,b,c) \in \mathbb{P}^2 - \mathfrak{D}$, namely $A_{a,b,c} = k\langle x,y,z\rangle$ modulo the relations

$$ax^2 + byz + czy = 0$$
$$ay^2 + bzx + cxz = 0$$
$$az^2 + bxy + cyx = 0$$

where $\mathcal{D} = \{12 \text{ explicit points}\}$.

Theorem (Artin-Tate-Van den Bergh)

If $(a,b,c) \in \mathbb{P}^2 - \mathfrak{D}$, then $A_{a,b,c}$ is 3-dim'l AS-regular. Its point scheme is an elliptic curve E naturally embedded in $\mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$ as the graph Γ_σ of a canonically determined $\sigma \in \mathrm{Aut}(E)$. Furthermore, there is a surjective map $\Phi : A \twoheadrightarrow B(E,\mathcal{L},\sigma)$ where $\mathcal{L} = i^*\mathrm{pr}_1^*\mathcal{O}(1)$ where $\Gamma_\sigma \xrightarrow{i} \mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{pr_1} \mathbb{P}^2$. Furthermore, $\ker\Phi = (g)$ where $g \in A_3$ is central.

Picture: $\mathrm{Proj}_{nc}(A_{a,b,c})$ is a nc analogue of $\mathbb{P}^2$ containing an elliptic curve E as a divisor. E is the zero locus of g.

4-dimensional Sklyanin algebras I

Let E be an elliptic curve and τ a translation automorphism that is not of order 1, 2, or 4.
Sklyanin defined a family of nc algebras $A(E,\tau) = k\langle x_0, x_1, x_2, x_3\rangle$ modulo 6 quadratic relations with structure constants depending on (E,τ).

Theorem (Stafford)

$A(E,\tau)$ is a 4-dim'l AS-regular algebra, a Koszul algebra, a noetherian domain, $H(A;t) = (1-t)^{-4}$, and there is a surjective homomorphism $\Phi : A(E,\tau) \twoheadrightarrow B(E,\mathcal{L},\tau)$ where $\mathcal{L}$ is an invertible $\mathcal{O}_E$-module of degree 4. Moreover, $\ker(\Phi) = (\Omega_1,\Omega_2)$ where $\Omega_1,\Omega_2 \in A_2$ are central.

Picture: $\operatorname{Proj}_{nc}(A(E,\tau))$ is a nc analogue of $\mathbb{P}^3$ containing a pencil of nc-quadrics, $Q_\lambda(E,\tau) := \operatorname{Proj}_{nc}(A/(\lambda_1\Omega_1+\lambda_2\Omega_2))$, $\lambda = (\lambda_1,\lambda_2) \in \mathbb{P}^1$, with base locus isomorphic to E. E is the zero locus of (Ω_1,Ω_2).

4-dimensional Sklyanin algebras II

The homogenization of the quantized enveloping algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is a degenerate 4-dimensional Sklyanin algebra. The 2-torsion subgroup of E behaves somewhat like a Weyl group. In some sense, $A(E,\tau)$ is an "elliptic homogenized version of $U(\mathfrak{sl}(2,\mathbb{C}))$".

Theorem (Van den Bergh)

Vague: there is a translation principle for $Q_\lambda(E,\tau)$ analogous to that for the primitive quotients $U(\mathfrak{sl}(2,\mathbb{C}))/(\Omega-\lambda)$ (see first talk).

Another analogy: the secant lines for $E \subset \mathbb{P}(A_1^*) = \mathbb{P}^3$ play the same role for $A(E,\tau)$ as the Verma modules for $U(\mathfrak{sl}(2,\mathbb{C})$ as the Borel subalgebra varies: the Verma modules provide rulings on the affine quadrics $\operatorname{Spec}_{nc}(U\big(\mathfrak{sl}(2,\mathbb{C}))/(\Omega-\lambda)\big)$.

The pencil $Q_\lambda(E,\tau)$ vs. a generic pencil of quadrics in $\mathbb{P}^3$

The commutative pencil:

A generic pencil of quadrics in $\mathbb{P}^3$ has exactly 4 singular members. Their common intersection is an elliptic curve, E say. The smooth quadrics have two rulings on them, and the singular ones have one. The lines on the quadrics are the secant lines to E.

The pencil of quadrics containing E may be labelled as Y_z, $z \in E/\pm = \mathbb{P}^1$, in such a way that

$$Y_z = \cup \text{ the secant lines } \overline{pq} \text{ such that } p+q=z.$$

The singular quadrics are Y_ω, $\omega \in E_2$, the 2-torsion subgroup of E. When $z \notin E_2$, the two rulings on Y_z are $\{\overline{pq} \mid p+q=z\}$ and $\{\overline{pq} \mid p+q=-z\}$, i.e., $Y_z = Y_{-z}$.

The non-commutative pencil:

There are exactly 4 singular $Q_\lambda(E,\tau)$ meaning $\operatorname{Ext}^n_{Q_\lambda}(-,-) \neq 0$ for all $n \geq 0$.

The common intersection of the $Q_\lambda(E,\tau)$s is E because $A/(\Omega_1,\Omega_2) \cong B(E,\mathcal{L},\tau)$.

The pencil $Q_\lambda(E,\tau)$

There is a bijection $\{$line modules for $A(E,\tau)\} \longleftrightarrow \{$secant lines to $E\}$: if $p,q \in E$ and $W \subset A_1$ is the subspace vanishing on $\overline{pq} \subset \mathbb{P}(A_1^*)$, then $L(\overline{pq}) := A/WA$ is a line module & there are no others.
If $z \in E$, there is $0 \neq \Omega(z) \in \mathbb{C}\Omega_1 + \mathbb{C}\Omega_2$ such that

$$L(\overline{pq})\Omega(z) = 0 \iff p+q \in \{z, -z-2\tau\}.$$

Relabel the $Q_\lambda(E,\tau)$s:

$$Q_z := \text{Proj}_{nc}\left(\frac{A(E,\tau)}{(\Omega(z))}\right), \qquad z \in E.$$

Thus $Q_z = Q_{-z-2\tau}$.

Theorem (Van den Bergh-S)

The singular quadrics are $Q_{\omega-\tau}$, $\omega \in E_2$. Equivalently, Q_z is singular $\iff$ it has two rulings.

Exponential growth

Example [Rogalski] The ring

$$A = \frac{k\langle x, y, z\rangle}{xy - yx = z^2, [x,z] = [y,z] = 0}.$$

is 3-dim'l AS-regular.

1. $\mathrm{GKdim}(A) = 3$
2. $\mathrm{Fract}_{\mathrm{gr}}(A) \cong D[t, t^{-1}]$ where $D = \mathrm{Fract}(\text{Weyl algebra})$
3. $D \supset$ a free subalgebra $k\langle f, g\rangle$ (Makar-Limanov)
4. $k[fz, gz, z]$ has exponential growth
5. $\mathrm{Fract}_{\mathrm{gr}}(k[fz, gz, z]) \subset \mathrm{Fract}_{\mathrm{gr}}(A)$ but $\mathrm{GKdim}\left(k[fz, gz, z]\right) = \infty$
6. $k[fz, gz, z]$ is not strongly noetherian

Proposition (Rogalski-Zhang)

Let σ be a non-quasi-unipotent automorphism of X. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module such that $\mathcal{L}_\sigma^n$ is very ample for some $n \geq 1$. Then $B(X, \mathcal{L}, \sigma)$ has exponential growth and is not noetherian.

nc tori and nc elliptic curves (Polishchuk & Schwarz)

T^2_θ := the nc 2-torus defined implicitly by declaring that the C*-algebra of continuous $\mathbb{C}$-valued functions on it is the universal C*-algebra A_θ generated by two unitary elements u and v such that

$$vu = e^{2\pi i\theta} uv.$$

$T^2_{\theta,\tau} := T^2_\theta$ with complex structure $(\mathcal{A}_\theta, \delta_\tau)$ where

$$\mathcal{A}_\theta := \left\{ \sum_{m,n} a_{mn} u^m v^n \in A_\theta \;\middle|\; (m,n) \mapsto a_{mn} \text{ is rapidly decreasing at } \infty \right\}$$

and $\delta_\tau : \mathcal{A}_\theta \to \mathcal{A}_\theta$ is the derivation $\delta_\tau(u) = u$ & $\delta_\tau(v) = \tau v$ (cf. $\partial_{\overline{z}}$).

$\mathfrak{C} := \mathfrak{coh}(T^2_{\theta,\tau})$ is defined via "holomorphic bundles" $(E, \overline{\nabla})$ on $T^2_{\theta,\tau}$

P's generalization of AZ Theorem $\Longrightarrow$ $B(\mathfrak{C}, \mathcal{O}, s)$ is a hcr for $T^2_{\theta,\tau}$.

$B(\mathfrak{C}, \mathcal{O}, s) = \mathbb{C}[B_1]$, $\dim(B_1) < \infty$, B is not noetherian but is graded-coherent.

"Degenerate" Sklyanin algebras

Suppose $k = \overline{k}$ has a primitive cube root of unity ω.

Assume $(a, b, c) \in \mathfrak{D}$

Theorem (C. Walton)

The Sklyanin algebra $A_{a,b,c}$ has infinite global dimension, is not noetherian, has exponential growth, and has zero divisors.

Theorem

1. *There is a quiver Q, independent of $(a, b, c) \in \mathfrak{D}$, such that $\mathfrak{QGr}(A_{a,b,c}) \equiv \mathfrak{QGr}(kQ) \equiv \mathfrak{Mod}S(Q)$ where $S(Q) = \varinjlim S_n$ and each S_n is a product of three matrix algebras.*
2. *There is an action of $\mu_3 = \sqrt[3]{1} \subset k^\times$ as automorphisms of the free algebra $F = k\langle X, Y\rangle$ s.t. $\mathfrak{QGr}(A_{a,b,c}) \equiv \mathfrak{QGr}(F \rtimes \mu_3)$.*

Entropy of non-noetherian hcrs

Newman-Schneider-Shalev:

the entropy of a locally finite $\mathbb{N}$-graded k-algebra A is

$$e(A) := \limsup_{n\to\infty} \sqrt[n]{\dim_k(A_n)} \quad = \frac{1}{\text{radius of conv. of } H(A;t)}.$$

$$\text{GKdim}(A) < \infty \quad \Longrightarrow \quad e(A) = 0 \text{ or } 1$$

Theorem (Stephenson-Zhang)

$e(A) > 1 \quad \Longrightarrow \quad$ *A is not noetherian.*

We say A has exponential growth if $e(A) > 1$

e.g.

$$e\big(k\langle x_1, \dots, x_d\rangle\big) = d$$

e(monomial alg.) $=$ exp(entropy in symbolic dynamics)

A tautology:

$$A = \text{a monomial algebra} \;=\; \frac{kQ}{(\text{finite } \# \text{ forbidden words})}$$

$\Longrightarrow$ $\log\big(e(A)\big) =$ the entropy of the dynamical system (X,σ) where

$$X = \{\text{bi-infinite legal words/paths}\} \subset \{\text{arrow}\}^{\mathbb{Z}}$$

discrete topology + product topology + subspace topology and

$$\sigma : X \to X$$

$$\sigma(\ldots a_{-1}a_0 \bullet a_1 a_2 \ldots) = \ldots a_0 a_1 \bullet a_2 \ldots$$

Proof: $\{$legal words of length $n\}$ = basis for A_n

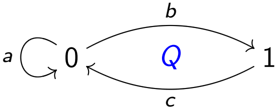
e.g., the Golden Mean Shift and $\mathbb{C}\langle x, y\rangle/(y^2)$

$X = \Big\{$sequences in $\{0,1\}^{\mathbb{Z}}\Big\}$ s.t. 1 must be followed by 0

$$\text{entropy}(X,\sigma) = \log\left(\frac{1+\sqrt{5}}{2}\right)$$

$$A := \mathbb{C}\langle x, y\rangle/(y^2) \qquad e(A) = \frac{1+\sqrt{5}}{2}$$

$A \hookrightarrow \mathbb{C}Q$



via $\begin{cases} x \mapsto a+b \\ y \mapsto c \end{cases}$

$$\mathfrak{QGr}(A) \equiv \mathfrak{QGr}(\mathbb{C}Q) \equiv \mathfrak{Gr}(S) \equiv \mathfrak{Mod}(S_0)$$

$S \xrightarrow[\text{subalg}]{\text{dense}}$ $\Bigg\{$ Cuntz-Krieger algebra for Q =
the graph C*-algebra for Q

$S_0 \xrightarrow[\text{subalg}]{\text{dense}}$ Connes's C*-algebra for the space of Penrose tilings

Remark: the Cuntz-Krieger algebra for a finite directed graph is a groupoid C*-algebra.

Zhang's 2-dim'l regular algebras with exponential growth

Theorem (Zhang)

Let $A = k[A_1]$ *with* $\dim(A_1) = n \geq 2$*. Then* A *is AS-regular (without the* $\text{GKdim} < \infty$ *condition) of global dimension 2* $\Longleftrightarrow$

$$A \;\cong\; \frac{k\langle x_1, \dots, x_n\rangle}{(x_1 \otimes y_1 + \cdots + x_n \otimes y_n)}$$

where $\operatorname{span}\{y_1, \dots, y_n\} = A_1$*. Furthermore,* $H(A;t) = (1 - nt + t^2)^{-1}$ *so* $e(A) = \frac{1}{2}(n + \sqrt{n^2 - 4})$*. Also,* A *is graded-coherent.*

Theorem (Van den Bergh, Minamoto, Piontkovski)

Let $A = k[A_1]$ *be a connected graded AS-regular algebra of global dimension 2. If* $\dim(A_1) = n$*, then*

$$\mathrm{D}^b\big(\mathfrak{Rep}(\,\bullet \xrightarrow{n \;\; \text{arrows}} \bullet\,)\big) \equiv \mathfrak{QGr}(A)$$

Kontsevich-Rosenberg's $\mathbb{P}^{n-1}_{nc}$

[KR] define the non-commutative projective space $\mathbb{P}^{n-1}_{nc}$ as the space representing the functor $\mathfrak{AffSch}_{nc} \to \mathfrak{Sets}$

$$A \mapsto \mathrm{Maps}\big(\mathsf{Spec}(A), \mathbb{P}^{n-1}_{nc}\big) :=$$

quotients M of $A^{\oplus n}$ s.t. $M \cong A$ locally in the flat topology.

Theorem (Kontsevich-Rosenberg)

The category $\mathfrak{Qcoh}(\mathbb{P}^{n-1}_{nc})$ *has cohomological dimension 1 and*

$$\mathsf{D}^b\big(\mathfrak{Qcoh}(\mathbb{P}^{n-1}_{nc})\big) \equiv \mathsf{D}^b\big(\mathsf{Rep}(\, \bullet \xrightarrow{n \;\; arrows} \bullet \,)\big)$$

A final remark

$$A := k[A_1] = \frac{k\langle x_1, \dots, x_n\rangle}{(x_1 \otimes y_1 + \cdots + x_n \otimes y_n)} \qquad \text{where } \operatorname{span}\{y_1, \dots, y_n\} = A_1.$$

Let $X = \operatorname{Proj}_{nc}(A)$ and $\mathcal{O}_X = \pi^* A_A$. Define $K_0(X) := K_0(\mathfrak{qgr}(A))$.

Theorem (Sisodia)

$$\big(K_0(X), K_0(X)^+, [\mathcal{O}_X]\big) \cong \big(\mathbb{Z}[\theta], \mathbb{R}^{\geq 0} \cap \mathbb{Z}[\theta], 1\big) \quad \textit{where} \quad \theta = \tfrac{n - \sqrt{n^2-4}}{2}.$$

If $n = 3$, $\mathbb{Z}[\theta] = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$.

Theorem (Pimsner-Voicelescu & Rieffel)

Let $\theta \in \mathbb{R} - \mathbb{Q}$ and A_θ the C^*-algebra of functions on the nc torus T_θ^2. Then

$$\big(K_0(A_\theta), K_0(A_\theta)^+, [A_\theta]\big) \cong \big(\mathbb{Z} + \mathbb{Z}\theta, \mathbb{R}^{\geq 0} \cap (\mathbb{Z} + \mathbb{Z}\theta), 1\big).$$

THE END