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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Vasily Dolgushev

Talk Title: Deformation Quantization I

Date: 01 / 29 / 13 **Time:** 3 : 45 am / pm (circle one)

List 6-12 key words for the talk: formal deformation, Maurer-Cartan (MC) element, Poisson structure, Kontsevich's Formality Theorem

Please summarize the lecture in 5 or fewer sentances: Introduce the Hochschild cochain complex of an associative algebra A and show that it "governs" the deformations of A in some sense. Provide some questions about construction and classification of the formal deformations of some examples. Introduce Maurer-Cartan elements and Kontsevich's formality theorem to tackle those questions.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
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01/29/13 Vasily Dolgushev: "Deformation Quantization I"
3:45pm

- let k be any field of characteristic 0
  $\otimes$, Hom will be over k unless specified otherwise
  $V^{\bullet}$ graded object
  $(sV)^{\bullet} := V^{\bullet-1}$ suspension, shift degree -1
  $(s^{-1}V)^{\bullet} = V^{\bullet+1}$
  for $v \in V$, $|v|$ is the degree of v

- let A be an associative algebra over k
  ε is a formal variable (deformation parameter)
  $m_{\varepsilon} : A \otimes A \to A[[\varepsilon]]$ such that
  $$m_{\varepsilon}(m_{\varepsilon}(a,b),c) = m_{\varepsilon}(a, m_{\varepsilon}(b,c)) \quad , \forall a,b,c \in A$$
  $$m_{\varepsilon}(a,b)|_{\varepsilon=0} = a \cdot b$$
  extend m_{ε} by $k[[\varepsilon]]$-linearity, we get a map
  $$m_{\varepsilon} : A[[\varepsilon]] \otimes_{k[[\varepsilon]]} A[[\varepsilon]] \longrightarrow A[[\varepsilon]]$$

  m_{ε} is a formal deformation of A
  $$m_{\varepsilon}(a,b) = a \cdot b + \sum_{t=1}^{\infty} \varepsilon^{t} m_t(a,b) \quad , \text{ where } m_t : A \otimes A \to A$$

- Hochschild cochain complex:
  $$C^{\bullet}(A) := \mathrm{Hom}(A^{\otimes \bullet}, A)$$
  $$A \to \mathrm{Hom}(A,A) \longrightarrow \mathrm{Hom}(A^{\otimes 2}, A) \longrightarrow \cdots$$
  let $P_1, P_2 \in C^{\bullet}(A)$, $P_i \in \mathrm{Hom}(A^{\otimes n_i}, A)$

$$P_1 \bullet P_2(a_1, a_2, \ldots, a_{n_1+n_2-1}) = (-1)^{|P_1|+1} \sum_{i \geq 0} (-1)^{i(|P_2|+1)} P_1(a_1, \ldots, a_i, P_2(a_{i+1}, \ldots), \ldots, a_{n_1+n_2-1})$$
$a_i \in A$

$$= \sum_{i \geq 0} (-1)^{|P_1|+1} (-1)^{i(|P_2|+1)} P_1(a_1, \ldots, a_i, P_2(a_{i+1}, \ldots), \ldots, a_{n_1+n_2-1})$$

$$[P_1, P_2]_G = P_1 \circ P_2 + (-1)^{|P_1||P_2|} P_2 \circ P_1$$

<u>Theorem.</u>: $[\ ,\ ]_G$ is a Lie bracket. (Gerstenhaber)

let $m \in Hom(A \otimes A, A)$

$$m \circ m(a, b, c) = m(m(a,b), c) - m(a, m(b,c))$$

<u>Prop.</u>: m defines an associative multiplication

$\Leftrightarrow m \circ m = 0$

$\Leftrightarrow [m, m]_G = 0.$

If $m_0 \in Hom(A \otimes A, A)$, $m_0(a,b) := a.b$, then

$$[m_0, m_0]_G = 0 \Rightarrow \partial = [m_0, -]_G$$
$$\Rightarrow \partial^2 = 0$$

$$m_\varepsilon : A \otimes A \to A[[\varepsilon]]$$
$$m_\varepsilon \in Hom(A \otimes A, A)[[\varepsilon]] = C^2(A)[[\varepsilon]]$$
$$m_\varepsilon = m_0 + \varepsilon m_1 + \varepsilon^2 m_2 + \cdots$$

Associative equation for m_ε can be written in the form

$[m_\varepsilon, m_\varepsilon]_G = 0$ in degrees in ε

$\varepsilon^0 : [m_0, m_0]_G = 0$

$\varepsilon^1 : [m_0, m_1]_G = 0$

$\Rightarrow m_1$ is a cocycle in $C^2(A)$

ε^2: $[m_0, m_2]_G + \frac{1}{2}[m_1, m_1]_G = 0$

$$\frac{1}{2}[m_1, m_1]_G = -\partial(m_2) \quad \text{in } H^*(C^\bullet(A)) \left[[m_1,m_1]_G = 0\right]$$

- let $A = k[x^1, x^2, x^3, \dots, x^d]$

<u>Examples</u>:

$$\|\lambda^{ij}\| \in Mat_{d\times d}(k)$$

$$m_\varepsilon(a,b) = \left(\exp\left(\varepsilon \lambda^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j}\right) a(x)\, b(y)\right)\Big|_{x=y}$$

m_ε is an <u>assoc. deformation</u> of $k[x^1, x^2, \dots, x^d]$. If $\lambda^{ij} = \lambda^{ji}$, then deformation is isom. to $Mat_{d\times d}(k)$

<u>Defn</u>: Two deformations m_ε, $\widetilde{m_\varepsilon}$ are equivalent if $\exists\, T: A \to A[[\varepsilon]]$ such that $T(a)|_{\varepsilon=0} = a$ and

$$T(\widetilde{m_\varepsilon}(a,b)) = m_\varepsilon(T(a), T(b))$$

$$T: A[[\varepsilon]] \longrightarrow A[[\varepsilon]]$$

- let $\mathfrak{g}$ be a finite dimensional Lie algebra over k

$\mathcal{U}(\mathfrak{g})$: universal enveloping algebra of $\mathfrak{g}$

PBW: $S(\mathfrak{g}) \xrightarrow{\text{sym}} \mathcal{U}(\mathfrak{g})$ is an isomorphism of vector spaces

$$\mathcal{U}(\mathfrak{g}) = k \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \mathfrak{g}^{\otimes 3} \oplus \cdots / (\text{relations})$$

obtains an obvious increasing filtration

If $v \in F^t\mathcal{U}(\mathfrak{g})$, $v =$ sum of tensor monomials of length $\leq t$

Now, $S(\mathfrak{g})[\varepsilon] \xrightarrow{\text{sym}_\varepsilon} \left(\bigoplus_{t\geq 0} \varepsilon^t F^t \mathcal{U}(\mathfrak{g})\right)[\varepsilon]$

let $v_1, v_2, \dots, v_t \in S^t(\mathfrak{g})$

$$\mathrm{Sym}_\varepsilon(v_1 v_2 \cdots v_t) = \frac{\varepsilon^t}{t!} \sum_{\sigma \in S_t} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(t)}$$

Sym_ε is an isomorphism of $k[\varepsilon]$-modules.
By using Sym_ε, we get an associative (non-comm.) multiplication on $S(\mathfrak{g})[\varepsilon]$, which can be easily extended to $S(\mathfrak{g})[[\varepsilon]]$
$S(\mathfrak{g}) \cong$ polynomial algebra on $\mathfrak{g}^*$

Recall. $[m_\varepsilon, m_\varepsilon] = 0$ for $A = k[x^1, \dots, x^d]$

– Poly. vector fields $V_A^\bullet = S_A(\underbrace{s\mathrm{Der}(A)}_{\text{suspension of derivation of } A})$

$V_A^\bullet$ is the algebra of poly vector fields on k^d, $\deg(x^i) = 0$
$V_A^\bullet = k[x^1, \dots, x^d, \theta_1, \dots, \theta_d]$ and $\deg(\theta_i) = 1$
where

$$\theta_i = s\frac{\partial}{\partial x^i} \qquad \text{with } \theta_i\theta_j = -\theta_j\theta_i$$

let $u : V_A^\bullet \longrightarrow C^\bullet(A)$ be defined as :
for $v = \sum_{i_1,\dots,i_n} v^{i_1\dots i_n}(x)\,\theta_{i_1}\cdots\theta_{i_n}$

$$u(v)(a_1,\dots,a_n) := \sum v^{i_n i_{n-1}\dots i_1}\,\partial_{x^{i_1}} a_1 \cdots \partial_{x^{i_n}} a_n$$

Theorem: (Hochschild- Kostant- Rosenberg)
For $A = k[x^1, \dots, x^d]$, the map u induces an isomorphism
$V_A^\bullet \cong H^\bullet(C^\bullet(A))$

The induced Lie bracket on $V_A^\bullet$ coincides with the Schouten bracket $[\ ,\ ]_S$.

$$[v, w_1 w_2]_S = [v, w_1]_S w_2 + (-1)^{(|v|+1)|w_1|} w_1 [v, w_2]_S .$$

Now, $m_\varepsilon = m_0 + \varepsilon m_1 + \varepsilon^2 m_2 + \dots$ for $A = k[x^1, \dots, x^d]$

$\partial m_1 = 0 \Rightarrow \exists\, \pi \in V_A^2$ such that $m_1 - \mathcal{U}(\pi_1) = \partial(\dots)$

WLOG, $m_1 = \mathcal{U}(\pi_1)$

$[m_1, m_1]_G$ is ∂-exact $\Rightarrow \mathcal{U}([\pi_1, \pi_1]_S) = 0$

$\Rightarrow [\pi_1, \pi_1]_S = 0$

π_1 is a polynomial Poisson structure on k^d.

$m_\varepsilon = \exp(\varepsilon \lambda^{ij} \dots)$

$\Rightarrow \pi_1 = -\frac{1}{2}(\lambda^{ij} - \lambda^{ji})\, \theta_i \theta_j$

For $\mathcal{U}(\mathfrak{g})$, we get the usual linear Poisson structure on $\mathfrak{g}^*$.

- <u>Questions</u>: 1) Given a Poisson structure π_1 on k^d, can we construct m_ε such that $m_1 = \mathcal{U}(\pi_1)$

  2) Can we describe equivalence classes of deformations m_ε for k^d?

- <u>Motivation in Quantum Mechanics</u>: we can think of this problem 1) as a fancy version of quantum mechanics.

- We can try to tackle these questions by using Kontsevich's Formality Theorem:

dg Lie algebra and Maurer-Cartan elements

$\mathcal{L} = \bigoplus_t \mathcal{L}^t$ with bracket $[\,,\,]$

$$\partial : \mathcal{L}^t \longrightarrow \mathcal{L}^{t+1}$$

"Example": $C^\bullet(A)$, $[\,,\,]_G$, $\partial = [m_0, -]_G$

<u>Defn</u>: A Maurer-Cartan (MC) element of $\mathcal{L}$ is $\alpha \in \mathcal{L}$ such that
$$\partial\alpha + \frac{1}{2}[\alpha,\alpha] = 0$$

<u>Ex</u>: $C^\bullet(A)$, where A is an assoc. algebra
$$\Pi = \varepsilon m_1 + \varepsilon^2 m_2 + \dots \quad \in \varepsilon C^2(A)[[\varepsilon]]$$
$$[m_\varepsilon, m_\varepsilon] = 0 \iff \partial\Pi + \frac{1}{2}[\Pi,\Pi]_G = 0$$

<u>Ex</u>: $V_A^\bullet$, $\Pi_1 \in V_A^2$, $[\Pi_1, \Pi_1]_S = 0$, this is an MC element.

- $\mathcal{L}$ is a dg Lie algebra
  $\mathcal{L}[[\varepsilon]]$ with ∂, $[\,,\,]$, by $k[[\varepsilon]]$-linearity
  $\varepsilon\mathcal{L}[[\varepsilon]]$ is a pro-nilpotent Lie algebra
  $\exp(\varepsilon\mathcal{L}^0[[\varepsilon]])$ is a pro-nilpotent Lie group

$\exp(\varepsilon\mathcal{L}^0[[\varepsilon]])$ acts on MC elements; $\alpha \in \mathcal{L}[[\varepsilon]]$
let $z \in \varepsilon\mathcal{L}^0[[\varepsilon]]$, then
$$\exp(z)(\alpha) = e^{[z,-]}\alpha - \frac{e^{[z,-]}-1}{[z,-]}(\partial z)$$

In the "example" $C^\bullet(A)$, Π is MC element
$$\begin{pmatrix}\text{formal assoc.}\\ \text{deformation } m_\varepsilon\end{pmatrix} \longleftrightarrow \begin{matrix}\text{MC elements}\\ \Pi \in \varepsilon C(A)[[\varepsilon]]\end{matrix}$$

<u>Prop</u>: Equivalence m_ε, $\widetilde{m}_\varepsilon$ corresponds to $\Pi \sim \widetilde{\Pi}$
$\Rightarrow$ Classifying equiv. classes of deformations is exactly classifying the orbits of $\exp(\varepsilon\mathcal{L}^0[[\varepsilon]])$ on $\mathcal{L}[[\varepsilon]]$.

(continued lecture II on 01/30/13)