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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: David Ben-Zvi

Talk Title: Introduction to D-modules I

Date: 01/30/13 **Time:** 9:15 am / pm (circle one)

List 6-12 key words for the talk: Fourier transformation, Weyl algebra, microlocal character, flat connection, deRham complex

Please summarize the lecture in 5 or fewer sentances: Introduce the basic definitions (algebraic D-modules over Weyl algebra, over algebraic varieties. Operations and characteristics of D-modules correspond to those of PDE. Grothendieck definition, flat connections, and deRham complex are also discussed in this talk.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

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- <u>Overhead</u>: Obtain a copy or use the originals and scan them
- <u>Blackboard</u>: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
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☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

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01/30/13 9:15 am

David Ben-Zvi: "Introduction to D-modules I"

- Weyl algebra $D_{\mathbb{A}^1} = \mathbb{C}\langle x,y\rangle / \langle yx - xy = 1\rangle$
realized by
  + mult. by x (position operator)
  + differentiation $\frac{\partial}{\partial x} = y$ (momentum operator)

- let $D_k = \mathbb{C}\langle x,y\rangle / \langle yx - xy = k\rangle$

$$D_k \cong \begin{cases} D, & \text{if } k \neq 0 \\ \mathbb{C}[x,y] = \mathcal{O}(\mathbb{A}^2), & \text{otherwise} \end{cases}$$
where $\mathbb{A}^2 = T^*\mathbb{A}^1$

Equivalently, filter $D = \bigcup D_{\leq n}$ by order in y

This is an algebra filtration with mult. $D_{\leq n} \cdot D_{\leq m} < D_{\leq n+m}$
$gr\, D = \mathbb{C}[x,y]$

- "Heisenberg uncertainty":
D has no finite dimensional modules
let x,y act on V with $[y,x] = 1$
$0 = Tr([y,x]) = Tr(1) = \dim V \Rightarrow \dim V = 0$

Where to find modules?
Obvious, $x, \frac{\partial}{\partial x}$ act on $\mathbb{C}[x]$ = poly. func. on $\mathbb{A}^1$
$= D/D\partial = D \cdot 1$

Another module? Take $C^{\infty}(\mathbb{A}^1)$ and look at $C^{-\infty}(\mathbb{A}^1)$

distributions, which can be mult. by x and differentiate $\frac{\partial}{\partial x}$.

let $f \in C^\infty$ be a function or distribution

$$C^\infty \supset D \cdot f = D/D \left(\begin{array}{c}\text{poly. differential eqns} \\ \text{satisfied by } f\end{array}\right)$$

$$D \cdot e^{\lambda x} = D/D(\partial - x)$$

exponential function

- let $M_{e^{\lambda x}} = De^{\lambda x} = D/D(\partial - x)$

Let take a look at the solution of this differential eqn.

$$\mathrm{Hom}_D(M_{e^{\lambda x}}, C^\infty) = \text{solutions of } (\partial f = \lambda f) = \mathbb{C}e^{\lambda x}$$

- let $\delta_0 \in C^\infty$, $x\delta_0 = 0$

$$M_{\delta_0} = D \cdot \delta_0 = D/Dx \cong \mathbb{C}[\partial] \cdot \delta_0 = \mathbb{C}[y] \cdot \delta_0$$

- Holomorphic function : is determined by differential eqns, it satisfies up to finite ambiguity : $\mathrm{Hom}_D(M_f, C^\infty)$ finite dim'l.

- Operations on D-modules $\longleftrightarrow$ algebraic analog of operations on functions

Fourier transform $\hat{f}(1) = \int f(x)e^{1x}\,dx$

$M \rightsquigarrow \mathbb{F}(M)$

$\mathbb{F}(M_f) = M_{\hat{f}}$

$D \circlearrowleft$ $x \mapsto y$, $y \mapsto -x$

transform

$$\mathbb{F}(M) = \begin{array}{c} M \\ \circlearrowleft \\ D \xrightarrow[i]{} D \end{array}$$

- If V is a vector space, $D_V\text{-mod} \xrightarrow{\mathbb{F}} D_{V^*}\text{-mod}$

through Fourier transform. This is a hint of microlocal character of D-modules.

↳ We can think of D-modules living on a plane instead of on a line, they have more symmetry and structure.

$\mathbb{F} \longleftrightarrow 90^\circ$ rotation of $T^*\mathbb{A}^1$ (cotangent bundle)

$$\begin{array}{ccc} & \underbrace{\mathbb{A}^1 \times \mathbb{A}^1}_{e^{tx}} & \\ \swarrow & & \searrow \\ \mathbb{A}^1 & & \mathbb{A}^1 \\ \ni & & \ni \\ f & & \hat{f} \end{array}$$

$M_{e^{tx}}$

$D(\mathbb{A}^1) \qquad D(\mathbb{A}^1)$

$\mathbb{F} : M \longmapsto M \otimes_D M_{e^{tx}}$

- Suppose X is a smooth variety over $\mathbb{C}$

  D_X is a sheaf of associative algebras on X

$$D = \langle \underset{\text{functions}}{\mathcal{O}}, \underset{\text{Vector field}}{T} \rangle \Big/ \left(\begin{array}{l} \partial f - f\partial = f' = \partial(f) \ \text{Leibniz rule} \\ \partial_1\partial_2 - \partial_2\partial_1 = [\partial_1, \partial_2] \end{array} \right)$$

Lie algebroid

↳ "$D = UT$"

- <u>Grothendieck definition</u>:

  $\mathrm{End}_{\mathbb{C}}\, \mathcal{O}_X$ can be considered as an $\mathcal{O}_X$-bimodule

  $\mathcal{O}_X$ = functions on X

  diff = differential = set theoretic support on Δ ← diagonal in $X \times X$

$$D_X = (\mathrm{End}_{\mathbb{C}}\, \mathcal{O}_X)^{\mathrm{diff}}$$

$$D = \bigcup (\mathrm{End}_{\mathbb{C}}\, \mathcal{O}_X)^{I^N}, \qquad I = \text{ideal of } \Delta \subset X \times X$$

I is generated by $f \otimes 1 - 1 \otimes f$, $f \in \mathcal{O}_X$

$I.L = 0$: $Lf = fL$

ie. $L \in \mathrm{End}_{\mathcal{O}}\, \mathcal{O} = \mathcal{O}$

when $N = 2$, $fL - Lf =: L(f) \in \mathcal{O}$

$D = \bigcup D_{\leq n}$

associate graded $\operatorname{gr} D = \operatorname{Sym} \mathcal{T} = \mathcal{O}(T^*X)$

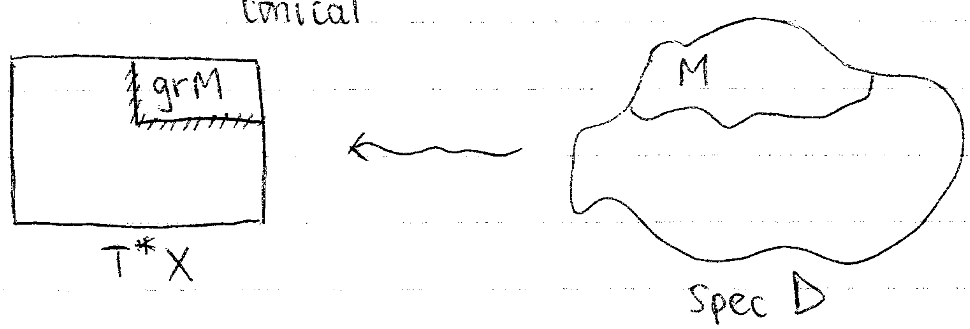
Visualize D microlocally (ie. via T^*X cotangent bundle of X)

e.g. M is a nice D-module, then we can find a good filtration

$M = \bigcup M_r \rightsquigarrow \operatorname{gr} M \circlearrowleft \mathcal{O}(T^*X)$

ie. extend M analog deformation $D \rightsquigarrow \mathcal{O}(T^*X)$

$\operatorname{gr} M$ is $\underbrace{\mathbb{C}^*\text{-equiv.}}_{\text{conical}}$ sheaf on T^*X



<u>Flat connections</u>:

let V be a vector space, ∇ is a vector bundle with flat connection

$$\nabla : V \to V \otimes \Omega^1$$

$$\nabla(fs) = f\,\nabla(s) + df\,s \quad , \quad \text{Leibniz rule}$$

$$\nabla_{[\partial_1, \partial_2]} = \nabla_{\partial_1}\nabla_{\partial_2} - \nabla_{\partial_2}\nabla_{\partial_1}$$

ie. $\mathcal{O} \circlearrowright V$ extends to $D \circlearrowright V$

$\mathcal{T} \circlearrowright V$, V is a D-module $\Rightarrow V, \nabla$ flat

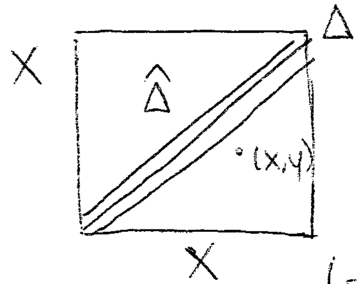
$$\xi \in \mathcal{T},\ \nabla_\xi : \begin{tikzcd} V \arrow[r, "\nabla"] \arrow[dr] & V \otimes \Omega^1 \arrow[d, "\xi"] \\ & V \end{tikzcd}$$

let V be a quasicoherent sheaf

V, ∇ flat $\longleftrightarrow D \circlearrowright V$

D-modules $\equiv$ q(uasi). coherent sheaf with flat connection

Grothendieck :



$\hat{\Delta} = (\widehat{X \times X})_{\Delta}$

$x \sim y \Leftrightarrow x, y$ infiniterminally nearby

(stratification $\leftrightarrow$ crystal)

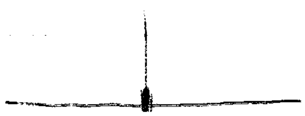
$D = (\mathcal{O}_{\hat{\Delta}})^*$ groupoid algebra

jets

Module for groupoid algebra : sheaf on X equivariant for equivalence relation

ie. sheaves on $X/\sim$

- Recall : $M_\delta = D \cdot \delta_0 = \mathbb{C}[\partial] \cdot \delta_0$



we cannot have finite dim'l D-module because this structure forces the modules to be infinite dimensional.

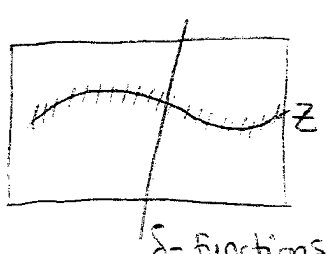
Corollary : Any $\mathcal{O}$-coherent D-module is a (flat) vector bundle.

Corollary : (Kashiwara's Thm).

$\underset{\text{smooth}}{Z} \xhookrightarrow{\text{closed}} \underset{\text{smooth}}{X}$, then there is an equivalence

$$(D_X\text{-mod})_Z \longleftrightarrow D_Z\text{-mod}$$

D_X- modules supported set theoretically on Z



parallel transport

- deRham complex:

a connection has a deRham complex flat $\nabla^2 = 0$

$$dR(M): M \xrightarrow{\nabla} M \otimes \Omega^1 \xrightarrow{\nabla} M \otimes \Omega^2 \to \cdots \to M \otimes \Omega^n$$

$H^0(dR(M)) = \{ s \in M : \nabla_\partial s = 0, \forall \partial \in \mathcal{T} \}$ flat sections

Now look at higher degree deRham cohomology

$$\begin{array}{ccccc} M \to M \otimes \Omega^1 \to \cdots \to & M \otimes \Omega^{n-1} & \longrightarrow & M \otimes \Omega^n \\ & \| & & \| \\ & M^r \otimes T & \longrightarrow & M^r \end{array}$$

$$H^n(dR(M)) = M^r / M^r T$$

- Where does $dR(M)$ live?

↳ $dR(M)$ is a dg module for (Ω^*, d)

In fact, we get an equivalence of derived categories

$$\underset{\text{reasonable}}{D_X\text{-mod}} \xrightarrow[dR(M)]{} (\Omega^*, d)$$

$$\begin{array}{ccc} D & & (\Omega^*, d) \\ \downarrow & & \downarrow \\ \mathrm{Sym}\, \mathcal{T} & & \Lambda^* \Omega^1 \end{array}$$

↳ $(\Omega^*, d) \simeq \mathbb{C}$ ($\Omega^* \to \mathbb{C}$ is a resolution)

dR: D-mod $\longrightarrow$ $\mathbb{C}$-sheaves $dR(M)$

$D \circlearrowright \mathcal{O} \longrightarrow \mathbb{C}$

We will look at the topological properties of D-modules.

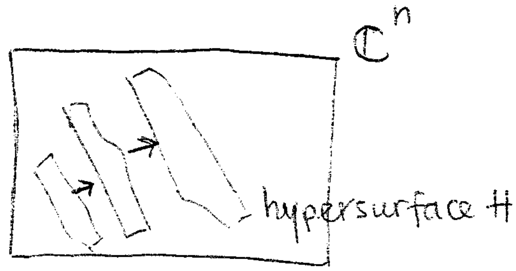
- Characteristics of a D-module $\longleftrightarrow$ characteristics of a PDE

<u>Note</u>: we want to look at D-module as a system of diff. eqns.

$$\underbrace{\mathcal{D}}_{\text{diff. eqns satisfied by } f} \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}\cdot f$$

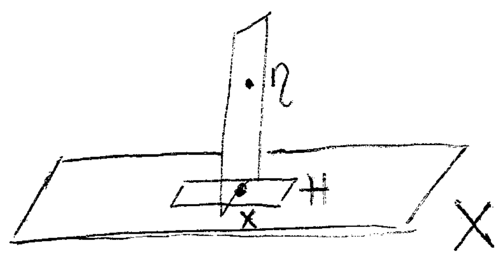
– <u>Cauchy-Kovalevskaya</u> (PDE "=" ODE)

Take initial values and move them, extend to a uniquely determined solution



D-module: $\eta \in T^*_x X$

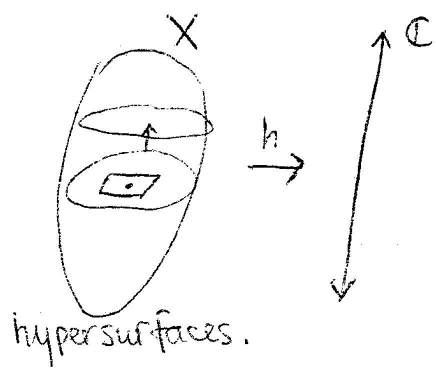
η is <u>noncharacteristic</u> if $dR(M)$ is locally constant transverse to H.



function $h: X \longrightarrow \mathbb{C}$

$$dh_x = \eta$$

we see if there is anything interesting happen when transform



<u>Theorem</u>: $\{\eta \in T^*X \text{ characteristic}\}$ = characteristic variety of M

$:= \operatorname{supp}(\operatorname{gr} M)$

(continued lecture II on 01/31/13)