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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: David Ben-Zvi

Talk Title: Introduction to D-modules II

Date: 01/31/13 **Time:** 9:15 am / pm (circle one)

List 6-12 key words for the talk: Holonomic module, perverse sheaf, pullback, pushforward, Riemann-Hilbert correspondence, Lagrangian Ch(M)

Please summarize the lecture in 5 or fewer sentances: Define holonomic modules and discuss functoriality of D-modules (pullback, pushforward). Discuss some of the varied perspectives on D-modules, in particular, the Beilinson-Bernstein localization theory for representations.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

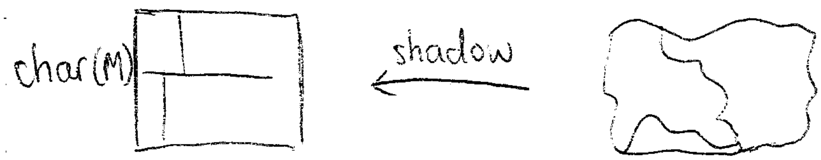
☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

01/31/13
9:15 am

<u>David Ben-Zvi</u> : "Introduction to D-modules II"

<u>Recall</u>:

cotangent bundle $T^*X \xleftrightarrow{h \leftrightarrow 0} D_X$-mod "quantum" modules on T^*X



For D-module M, characteristic char(M) is :

+ support of gr(M)
+ characteristics of PDE of M
+ Morse theory

<u>Defn</u>: M is <u>holonomic</u> if dim (char M) is minimal

$\Rightarrow$ dim $= n =$ dim X (by Bernstein)

In. fact $\Leftrightarrow$ char(M) $\subset T^*X$ (Gabber) is Lagrangian.

– Char M , $\mathbb{C}^*$- invt. Lagrangian $\subset T^*M$

$\quad ||$

$\coprod_{Y \text{ shadow}} T^*_Y X$, union of conormal bundles to subvarieties of X

If $Y = X$, $T^*_X X =$ 0-section , char(V, ∇) flat vector bundle

$Y = pt$, $T^*_x X =$ cotangent fiber , char (M_δ)

In fact, M is holonomic $\Rightarrow$ dR(M), complex of $\mathbb{C}$-sheaves , is a <u>constructible complex</u>.

H^i(dR(M)) are locally constant on each stratum.

Restrict to each stratum, $dR(M)|_Y$ captures monodromy of solutions of M

$\Rightarrow$ $dR(M)$ is a perverse sheaf

<u>Defn:</u> A <u>perverse sheaf</u> is $dR(M)$ when M is holonomic.

$$\begin{tikzcd} \text{Holonomic D-mod} \arrow[r, "dR"] & \text{constant complexes} \arrow[l, dashed, bend right] \arrow[dl, "\sim"] \\ \text{regular holonomic} \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

(by Kashinawa - Mebkhovl)] Riemann-Hilbert correspondence

<u>Defn:</u> M is <u>regular</u> if formal series solutions (along curves) converge.

- <u>Functoriality of D-modules</u>:
  Functions pull back $\quad X \xrightarrow{\pi} Y \quad \rightsquigarrow \pi^*: C^\infty(Y) \to C^\infty(X)$
  Distributions push forward $\quad \rightsquigarrow \pi_*: C^{-\infty}(X) \to C^{-\infty}(Y)$
  D-modules do both.

+ <u>Pull back</u>: $\pi^!: D_Y\text{-mod} \longrightarrow D_X\text{-mod}$ is just pullback for flat vector bundles / quasi-coherent sheaves.

+ <u>Push forward</u>: $i: Z \hookrightarrow X$ closed embedding
  $i_*: D_Z\text{-mod} \longrightarrow D_X\text{-mod}$

- let π be submersion/smooth : "deRham cohomology along fiber"

$$\begin{tikzcd} \pi: X \arrow[r] \arrow[dr, "\Gamma_\pi"'] & Y \\ & X \times Y \arrow[u, "p"'] \end{tikzcd}$$

$$M \to M \otimes \Omega^1_{/Y} \to \cdots \to M \otimes \Omega^{top}_{/Y}$$

ie. fiber by fiber $\pi_* M|_y$ is $H^*(p^{-1}(y), dR(M)_{\pi^{-1}y})$

ie, solutions of $\pi_* M$ given by integrals over cycles in fiber of solutions to M: $\int_{c_i} f\omega_i$

↳ This is a source of integral rep. of special functions.

ex: $y^2 = x(x-1)(x-\lambda)$

$$\int_B \omega = \int \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \rightsquigarrow \pi_* \mathcal{O}$$

- Picture: a (regular) holonomic D-module on X is an avatar of a space

$$\begin{array}{ccc} Z & \xrightarrow{p} & X \\ \cup & & \cup \\ Z^\circ & \longrightarrow & X^\circ \end{array}, \quad p_* \mathcal{O} \text{ is a regular holonomic D-mod on } X$$

Z° is subspace where p is smooth, then $p_* \mathcal{O}$ is a flat vector bundle on X°

- Functors R-mod $\longrightarrow$ S-mod are given by bimodules

$$M_R \mapsto M \otimes_R B_S$$

- Functors on D-modules:

$f: X \to Y$,

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

$$T^*(X \times Y) \supset Z_f = \text{conormal to } \Gamma_f \subset T^*(X \times Y)$$

$$\begin{array}{ccc} & Z_f & \\ \swarrow & & \searrow \\ T^*X & & T^*Y \end{array}$$

$$\text{so } D_X\text{-mod} \underset{f_*}{\overset{f^!}{\rightleftarrows}} D_Y\text{-mod}$$

$$\begin{array}{ccc} & D_{X \to Y} & \\ D_X & & D_Y \end{array}$$

If $\begin{tikzcd} & W \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$ then $D_X\text{-mod} \xrightarrow{p_{Y*} p_X'} D_Y\text{-mod}$, is a functor on D-modules

- How do we compose correspondences / functors?
  [Goncharov], we do base change

$$\begin{tikzcd} & & W_Y \times Z \arrow[dl] \arrow[dr] & & \\ & W \arrow[dl] \arrow[dr] & & Z \arrow[dl] \arrow[dr] & \\ X & & Y & & U \end{tikzcd}$$

ex: $\begin{tikzcd} & W \arrow[dl, "p_X"'] \arrow[d, "a"] \arrow[dr, "p_Y"] & \\ X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$

$p_{Y*} p_X'$

$\pi_{2*}(\pi_1'(-) \otimes K)$

where $a_* \mathcal{O}_W =: K$

- Beilinson - Bernstein:
  Naturally we can think of D "=" $U\mathcal{T}$
  If $G \circlearrowright X$ action, $\mathfrak{g}$ is Lie algebra, then
  $\mathfrak{g} \longrightarrow \Gamma(\mathcal{T}_X)$
  $U\mathfrak{g} \longrightarrow \Gamma(D_X)$ where $U\mathfrak{g}$ = univ. enveloping alg. of $\mathfrak{g}$

+ Write differential eqns using elements of $U\mathfrak{g}$
+ If $D_X \circlearrowright M$, then $\Gamma(D_X) \circlearrowright \Gamma(M)$ is a representation of $\mathfrak{g}$

$$\begin{array}{ccc} \Gamma(D_X) & \circlearrowright & \Gamma(M) \\ \cup & \circlearrowleft & \\ U\mathfrak{g} & & \end{array}$$

$$\Delta : U\mathfrak{g}\text{-mod} \rightleftarrows D_X\text{-mod} : \Gamma$$
$$N \longmapsto N \otimes_{U\mathfrak{g}} D_X$$

where $\Gamma = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, M) = \mathrm{Hom}_D(D, M)$

If $G \circlearrowright X$, then $G \circlearrowright T^*X$ $\quad$ $\mathrm{Sym}\,\mathcal{T}$

$$\begin{array}{c} T^*X \\ \downarrow \mu \\ \mathfrak{g}^* \end{array} \qquad \mathrm{Sym}\,\mathfrak{g}$$

$$\mu^* : \mathrm{Sym}\,\mathfrak{g}\text{-mod} \leftrightarrows \mathrm{Sym}\,\mathcal{T}\text{-mod} : \mu_*$$

- If $X = G/B$, where G = complex reductive $(GL_n(\mathbb{C}))$
  B = Borel $(\triangledown)$
  $\searrow$ X is a full flag in $\mathbb{C}^n$

$T^*G/B = \{x \in \mathfrak{g},\ B' \ni x \text{ nilpotent}\}$ flag perserved strictly by x

$$\begin{array}{c} T^*G/B \\ \text{Springer resolution}\ \big\downarrow \\ \mathfrak{g}^* \supset \underset{\text{nilpotent}}{N} \end{array}$$

<u>ex</u>: $T^*\mathbb{P}^1$
$\downarrow$
 $\subset \mathfrak{sl}_2$

- $G \circlearrowright G/B$

where

$$I_0 \subset Z \subset \mathcal{U}\mathfrak{g}, \qquad Z \cong \mathbb{C}[\mathfrak{h}^*/W]$$

$$\begin{array}{ccc} U\mathfrak{g} & \longrightarrow & \Gamma(D) \\ & \searrow & \wr\| \\ & & \mathcal{U}\mathfrak{g}/I_0 \end{array}$$

where W = Weyl group, $\mathfrak{h}^*$ = dual Cartan

<u>Theorem.</u>: (Beilinson - Bernstein)

$$\Delta : \mathcal{U}\mathfrak{g}/I_0\text{-mod} \xrightarrow{\simeq} D_{G/B}\text{-mod} : \Gamma$$

$\mathcal{U}\mathfrak{g}/I_0$-mod $=$ $\mathfrak{g}$-reps with trivial central character

- <u>Classes of $\mathcal{U}\mathfrak{g}$-modules</u> we like to consider:
  $(\mathfrak{g}, K)$-modules, where $K \subset G$, Lie algebra $K \subset \mathfrak{g}$
  $K = B$ or N

or $K = G^\sigma$, "$G_{\mathbb{R}}$ real form", σ is an involution

- If K has finitely many orbits on G/B, then $(\mathfrak{g}, K)$-modules give regular holonomic D-modules (stratification by K-orbits)
