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NOTETAKER CHECKLIST FORM

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Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Matilde Marcolli

Talk Title: Noncommutative motives and their applications I

Date: 01 / 31 / 13 **Time:** 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: motives, Tannakian category, noncommutative (Chow) motives, motivic Galois groups

Please summarize the lecture in 5 or fewer sentences: Define different types of motives and discuss categories of noncommutative motives, their properties, and their relation to the classical (commutative) theory of motives. Introduce some main results in joint work with Goncalo Tabuada.

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Noncommutative motives and their applications

Matilde Marcolli and Goncalo Tabuada

MSRI 2013

The classical theory of pure motives (Grothendieck)

• $\mathscr{V}_k$ category of smooth projective varieties over a field k; morphisms of varieties

• (Pure) Motives over k: linearization and idempotent completion (+ inverting the Lefschetz motive)

• Correspondences: $\mathrm{Corr}_{\sim,F}(X,Y)$: F-linear combinations of algebraic cycles $Z \subset X \times Y$ of codimension = dim X

• composition of correspondences:

$$\mathrm{Corr}(X,Y) \times \mathrm{Corr}(Y,Z) \to \mathrm{Corr}(X,Z)$$

$$(\pi_{X,Z})_*(\pi_{X,Y}^*(\alpha) \bullet \pi_{Y,Z}^*(\beta))$$

intersection product in $X \times Y \times Z$

• Equivalence relations on cycles: rational, homological, numerical

- $\alpha \sim_{rat} 0$ if $\exists\ \beta$ correspondence in $X \times \mathbb{P}^1$ with $\alpha = \beta(0) - \beta(\infty)$ (moving lemma; Chow groups; Chow motives)
- $\alpha \sim_{hom} 0$: vanishing under a chosen Weil cohomology functor H^*
- $\alpha \sim_{num} 0$: trivial intersection number with every other cycle

The category of motives has different properties depending on the choice of the equivalence relation on correspondences

Effective motives Category $\mathrm{Mot}^{eff}_{\sim,F}(k)$:

• Objects: (X,p) smooth projective variety X and idempotent $p^2 = p$ in $\mathrm{Corr}_{\sim,F}(X,X)$

• Morphisms:

$$\mathrm{Hom}_{\mathrm{Mot}^{eff}_{\sim,F}(k)}((X,p),(Y,q)) = q\mathrm{Corr}_{\sim,F}(X,Y)p$$

• tensor structure $(X,p)\otimes(Y,q) = (X\times Y, p\times q)$

• notation $h(X)$ for motive (X, id)

Tate motives

• $\mathbb{L}$ Lefschetz motive: $h(\mathbb{P}^1) = 1\oplus\mathbb{L}$ with $1 = h(Spec(k))$.

• formal inverse $\mathbb{L}^{-1}$ = Tate motive; notation $\mathbb{Q}(1)$

Motives Category $\mathrm{Mot}_\sim(k)$

- Objects: $(X,p,m) := (X,p)\otimes \mathbb{L}^{-m} = (X,p)\otimes \mathbb{Q}(m)$
- Morphisms:

$$\mathrm{Hom}_{\mathrm{Mot}_\sim(k)}((X,p,m),(Y,q,n)) = q\mathrm{Corr}^{m-n}_{\sim,F}(X,Y)p$$

shifts the codimension of cycles (Tate twist)

- Chow motives; homological motives; numerical motives

Jannsen's semi-simplicity result

Theorem (Jannsen 1991): TFAE

- $\mathrm{Mot}_{\sim,F}(k)$ is a semi-simple *abelian* category
- $\mathrm{Corr}^{\dim X}_{\sim,F}(X,X)$ is a finite-dimensional semi-simple F-algebra, for each X
- The equivalence relation is numerical: $\sim=\sim_{num}$

Known that with $\sim_{rat} \neq \sim_{num}$: category of Chow motives is *not* abelian
What about homological equivalence?

Weil cohomologies $H^* : \mathscr{V}_k^{op} \to VecGr_F$

- Künneth formula: $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$
- $\dim H^2(\mathbb{P}^1) = 1$; Tate twist: $V(r) = V \otimes H^2(\mathbb{P}^1)^{\otimes -r}$
- trace map (Poincaré duality) $tr : H^{2d}(X)(d) \to F$
- cycle map $\gamma_n : \mathscr{Z}^n(X)_F \to H^{2n}(X)(n)$ (algebraic cycles to cohomology classes)

Examples: deRham, Betti, ℓ-adic étale

Grothendieck's idea of motives: universal cohomology theory for algebraic varieties lying behind all realizations via Weil cohomologies

Grothendieck's standard conjectures

• (Künneth) C: The Künneth components of the diagonal Δ_X are algebraic

• (Hom=Num) D Homological and numerical equivalence coincide

• (Lefschetz) B: the Lefschetz involution $\star_{L,X}$ is algebraic ($\mathbb{Q}$-coeffs): $\star_{L,X}$ is L^{d-i} on $\oplus_{i,r} H^i(X)(r)$ for $i \leq d$ (inverse for $i > d$)

$$L^{d-i} : H^i(X)(r) \to H^{2d-i}(X)(d-i+r)$$

determined by hyperplane sections

• (Hodge) I The quadratic form defined by the Hodge involution $\star_H$ is positive definite on algebraic cycles with homological equivalence

We will focus later on C and D (in char 0 $B \Rightarrow$ all; $B + I \Rightarrow D$)

Motivic Galois groups

More structure than abelian category: Tannakian category $\mathrm{Rep}_F(G)$ fin dim lin reps of an affine group scheme G

- F-linear, abelian, tensor category (*symmetric monoidal*) $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$
- functorial isomorphisms:

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \stackrel{\simeq}{\to} (X \otimes Y) \otimes Z$$

$$c_{X,Y} : X \otimes Y \stackrel{\simeq}{\to} Y \otimes X \quad \text{with} \quad c_{X,Y} \circ c_{Y,X} = 1_{X \otimes Y}$$

$$\ell_X : X \otimes 1 \stackrel{\simeq}{\to} X, \quad r_X : 1 \otimes X \stackrel{\simeq}{\to} X$$

- *Rigid*: duality $\vee : \mathscr{C} \to \mathscr{C}^{op}$ with $\epsilon : X \otimes X^\vee \to 1$ and $\eta : 1 \to X^\vee \otimes X$

$$X \simeq X \otimes 1 \stackrel{1_X \otimes \eta}{\to} X \otimes X^\vee \otimes X \stackrel{\epsilon \otimes 1_X}{\to} 1 \otimes X \simeq X$$

composition is identity

• categorical trace (Euler characteristic)
$tr(f) = \epsilon \circ c_{X^\vee \otimes X} \circ (1_{X^\vee} \otimes f) \circ \eta$; $\dim X = tr(1_X)$

• Tannakian: as above (and with $\mathrm{End}(1) = F$) and *fiber functor*
$\omega : \mathscr{C} \to Vect(K)$
$K =$ extension of F; ω exact faithful tensor functor; *neutral Tannakian* if $K = F$

• equivalence $\mathscr{C} \simeq \mathrm{Rep}_F(G)$, affine group scheme
$G = \mathrm{Gal}(\mathscr{C}) = \underline{Aut}^{\otimes}(\omega)$

• Deligne's characterization (char 0): Tannakian iff $tr(1_X)$ non-negative for all X

Tannakian categories and standard conjectures

In the case of $\mathrm{Mot}_{\sim_{num}}(k)$, when Tannakian?

• problem: $tr(1_X) = \chi(X)$ Euler characteristic can be negative

• $\mathrm{Mot}^{\dagger}_{\sim_{num}}(k)$ category $\mathrm{Mot}_{\sim_{num}}(k)$ with modified commutativity constraint $c_{X,Y}$ by the Koszul sign rule
(corrects for signs in the Euler characteristic)

• (Jannsen) if standard conjecture C (Künneth) holds then $\mathrm{Mot}^{\dagger}_{\sim_{num}}(k)$ is Tannakian

• If conjecture D also holds then H^* fiber functor

Motives and Noncommutative motives

• Motives (pure): smooth projective algebraic varieties X
cohomology theories $H_{dR}, H_{Betti}, H_{etale}, \ldots$
universal cohomology theory: motives $\Rightarrow$ realizations

• NC Motives (pure): smooth proper dg-categories $\mathscr{A}$
homological invariants: K-theory, Hochschild and cyclic cohomology
universal homological invariant: NC motives

dg-categories

$\mathscr{A}$ category whose morphism sets $\mathscr{A}(x, y)$ are complexes of k-modules (k = base ring or field) with composition satisfying Leibniz rule

$$d(f \circ g) = df \circ g + (-1)^{\deg(f)} f \circ dg$$

$\mathtt{dgcat}$ = category of (small) dg-categories with dg-functors (preserving dg-structure)

From varieties to dg-categories

$$X \Rightarrow \mathscr{D}^{dg}_{perf}(X)$$

dg-category of perfect complexes

H^0 gives derived category $\mathscr{D}_{perf}(X)$ of perfect complexes of $\mathscr{O}_X$-modules

saturated dg-categories (Kontsevich)

- smooth dgcat: perfect as a bimodule over itself
- proper dgcat: if the complexes $\mathscr{A}(x,y)$ are perfect
- saturated = smooth + proper

smooth projective variety $X \Rightarrow$ smooth proper dgcat $\mathscr{D}^{dg}_{perf}(X)$
(but also smooth proper dgcat not from smooth proj varieties)

derived Morita equivalences

• $\mathscr{A}^{op}$ same objects and morphisms $\mathscr{A}^{op}(x,y) = \mathscr{A}(y,x)$; right dg $\mathscr{A}$-module: dg-functor $\mathscr{A}^{op} \to \mathscr{C}_{dg}(k)$ (dg-cat of complexes of k-modules); $\mathscr{C}(\mathscr{A})$ cat of $\mathscr{A}$-modules; $\mathscr{D}(\mathscr{A})$ (derived cat of $\mathscr{A}$) localization of $\mathscr{C}(\mathscr{A})$ w/ resp to quasi-isom

• functor $F : \mathscr{A} \to \mathscr{B}$ is derived Morita equivalence iff induced functor $\mathscr{D}(\mathscr{B}) \to \mathscr{D}(\mathscr{A})$ (restriction of scalars) is an equivalence of triangulated categories

• cohomological invariants (K-theory, Hochschild and cyclic cohomologies) are derived Morita invariant: send derived Morita equivalences to isomorphisms symmetric monoidal category $\mathtt{Hmo}$

• homotopy category: dg-categories up to derived Morita equivalences

• $\otimes$ extends from k-algebras to dg-categories

• can be derived with respect to derived Morita equivalences (gives symmetric monoidal structure on $\mathtt{Hmo}$)

• saturated dg-categories = dualizable objects in $\mathtt{Hmo}$ (Cisinski–Tabuada)

• Euler characteristic of dualizable object: $\chi(\mathscr{A}) = HH(\mathscr{A})$ Hochschild homology complex (Cisinski–Tabuada)

Further refinement: $\mathtt{Hmo}_0$

• all cohomological invariants listed are "additive invariants":

$$E : \mathtt{dgcat} \to A, \quad E(\mathscr{A}) \oplus E(\mathscr{B}) = E(|M|)$$

where A additive category and $|M|$ dg-category $Obj(|M|) = Obj(\mathscr{A}) \cup Obj(\mathscr{B})$ morphisms $\mathscr{A}(x, y)$, $\mathscr{B}(x, y)$, $X(x, y)$ with X a $\mathscr{A}$–$\mathscr{B}$ bimodule

• $\mathtt{Hmo}_0$: objects dg-categories, morphisms $K_0\mathrm{rep}(\mathscr{A}, \mathscr{B})$ with $\mathrm{rep}(\mathscr{A}, \mathscr{B}) \subset \mathscr{D}(\mathscr{A}^{op} \otimes^{\mathbb{L}} \mathscr{B})$ full triang subcat of $\mathscr{A}$–$\mathscr{B}$ bimodules X with $X(a, -) \in \mathscr{D}_{perf}(\mathscr{B})$; composition = (derived) tensor product of bimodules

• (Tabuada) $\mathscr{U}_A : \mathtt{dgcat} \to \mathtt{Hmo}_0$, id on objects, sends dg-functor to class in Grothendieck group of associated bimodule ($\mathscr{U}_A$ characterized by a universal property)

• all additive invariants factor through $\mathtt{Hmo}_0$

noncommutative Chow motives (Kontsevich) $\mathrm{NChow}_F(k)$

• $\mathtt{Hmo}_{0;F}$ = the F-linearization of additive category $\mathtt{Hmo}_0$

• $\mathtt{Hmo}_{0;F}^{\natural}$ = idempotent completion of $\mathtt{Hmo}_{0;F}$

• $\mathrm{NChow}_F(k)$ = idempotent complete full subcategory gen by saturated dg-categories

$\mathrm{NChow}_F(k)$:

- Objects: $(\mathscr{A}, e)$ smooth proper dg-categories (and idempotents)
- Morphisms $K_0(\mathscr{A}^{op} \otimes_k^{\mathbb{L}} \mathscr{B})_F$ (correspondences)
- Composition: induced by derived tensor product of bimodules

relation to commutative Chow motives (Tabuada):

$$\mathrm{Chow}_{\mathbb{Q}}(k)/_{-\otimes\mathbb{Q}(1)} \hookrightarrow \mathrm{NChow}_{\mathbb{Q}}(k)$$

commutative motives embed as noncommutative motives after moding out by the Tate motives

orbit category $\mathrm{Chow}_{\mathbb{Q}}(k)/_{-\otimes\mathbb{Q}(1)}$

$(\mathscr{C}, \otimes, \mathbf{1})$ additive, $F-$*linear*, rigid symmetric monoidal;
$\mathscr{O} \in \mathrm{Obj}(\mathscr{C})$ $\otimes$-invertible object:
orbit category $\mathscr{C}/_{-\otimes\mathscr{O}}$ same objects and morphisms

$$\mathrm{Hom}_{\mathscr{C}/_{-\otimes\mathscr{O}}}(X, Y) = \oplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X, Y \otimes \mathscr{O}^{\otimes j})$$

Numerical noncommutative motives

M.M., G.Tabuada, *Noncommutative motives, numerical equivalence, and semi-simplicity*, arXiv:1105.2950

$(\mathscr{A}, e)$ and $(\mathscr{B}, e')$ objects in $\mathrm{NChow}_F(k)$ and correspondences

$$\underline{X} = e \circ [\sum_i a_i X_i] \circ e', \qquad \underline{Y} = e' \circ [\sum_j b_j Y_j] \circ e$$

X_i and Y_j bimodules

$\Rightarrow$ intersection number:

$$\langle \underline{X}, \underline{Y} \rangle = \sum_{ij} [HH(\mathscr{A}; X_i \otimes^{\mathbb{L}}_{\mathscr{B}} Y_j)] \in K_0(k)_F$$

with $[HH(\mathscr{A}; X_i \otimes^{\mathbb{L}}_{\mathscr{B}} Y_j)]$ class in $K_0(k)_F$ of Hochschild homology complex of $\mathscr{A}$ with coefficients in the $\mathscr{A}$–$\mathscr{A}$ bimodule $X_i \otimes^{\mathbb{L}}_{\mathscr{B}} Y_j$

numerically trivial: $\underline{X}$ if $\langle \underline{X}, \underline{Y} \rangle = 0$ for all $\underline{Y}$

- $\otimes$-ideal $\mathscr{N}$ in the category $\mathrm{NChow}_F(k)$
- $\mathscr{N}$ largest $\otimes$-ideal strictly contained in $\mathrm{NChow}_F(k)$

numerical motives: $\mathrm{NNum}_F(k)$

$$\mathrm{NNum}_F(k) = \mathrm{NChow}_F(k)/\mathscr{N}$$

abelian semisimple (M.M., G.Tabuada, arXiv:1105.2950)

- $\mathrm{NNum}_F(k)$ is abelian semisimple

analog of Jannsen's result for commutative numerical pure motives

What about Tannakian structures and motivic Galois groups?

For commutative motives this involves standard conjectures (C = Künneth and D = homological and numerical equivalence)

Questions:

- is $\mathrm{NNum}_F(k)$ (neutral) super-Tannakian?
- is there a good analog of the standard conjecture C (Künneth)?
- does this make the category Tannakian?
- is there a good analog of standard conjecture D (numerical = homological)?
- does this neutralize the Tannakian category?
- relation between motivic Galois groups for commutative and noncommutative motives?

Tannakian categories $(\mathscr{C}, \otimes, \mathbf{1})$

F-linear, abelian, rigid symmetric monoidal with $\mathrm{End}(\mathbf{1}) = F$

- Tannakian: $\exists$ K-valued *fiber functor*, K field ext of F: exact faithful $\otimes$-functor $\omega : \mathscr{C} \to \mathrm{Vect}(K)$; neutral if $K = F$

$\omega \Rightarrow$ equivalence $\mathscr{C} \simeq \mathrm{Rep}_F(\mathrm{Gal}(\mathscr{C}))$ affine group scheme (Galois group) $\quad \mathrm{Gal}(\mathscr{C}) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$

- intrinsic characterization (Deligne): F char zero, $\mathscr{C}$ Tannakian iff $\mathrm{Tr}(id_X)$ non-negative integer for each object X

super-Tannakian categories $(\mathscr{C}, \otimes, \mathbf{1})$

F-linear, abelian, rigid symmetric monoidal with $\mathrm{End}(\mathbf{1}) = F$

$s\mathrm{Vect}(K)$ super-vector spaces $\mathbb{Z}/2\mathbb{Z}$-graded

- super-Tannakian: $\exists$ K-valued *super fiber functor*, K field ext of F: exact faithful $\otimes$-functor $\omega : \mathscr{C} \to s\mathrm{Vect}(K)$; neutral if $K = F$

$\omega \Rightarrow$ equivalence $\mathscr{C} \simeq \mathrm{Rep}_F(s\mathrm{Gal}(\mathscr{C}), \epsilon)$ super-reps of affine super-group-scheme (super-Galois group)
$s\mathrm{Gal}(\mathscr{C}) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ $\quad \epsilon =$ parity automorphism

- intrinsic characterization (Deligne) F char zero, $\mathscr{C}$ super-Tannakian iff Shur finite (if F alg closed then neutral super-Tannakian iff Schur finite)

- Schur finite: symm grp S_n, idempotent $c_\lambda \in \mathbb{Q}[S_n]$ for partition λ of n (irreps of S_n), Schur functors $S_\lambda : \mathscr{C} \to \mathscr{C}$, $\quad S_\lambda(X) = c_\lambda(X^{\otimes n})$
$\mathscr{C} =$ Schur finite iff all objects X annihilated by some Schur functor $S_\lambda(X) = 0$

Main results

M.M., G.Tabuada, *Noncommutative numerical motives, Tannakian structures, and motivic Galois groups*, arXiv:1110.2438

assume either: (i) $K_0(k) = \mathbb{Z}$, F is k-algebra; (ii) k and F both field extensions of a field K

- **Thm 1**: $\mathrm{NNum}_F(k)$ is super-Tannakian; if F alg closed also neutral
- **Thm 2**: standard conjecture $C_{NC}(\mathscr{A})$: the Künneth projectors

$$\pi^{\pm}_{\mathscr{A}} : \overline{HP}_*(\mathscr{A}) \twoheadrightarrow \overline{HP}^{\pm}_*(\mathscr{A}) \hookrightarrow \overline{HP}_*(\mathscr{A})$$

are algebraic: $\pi^{\pm}_{\mathscr{A}} = \overline{HP}_*(\underline{\pi}^{\pm}_{\mathscr{A}})$ with $\underline{\pi}^{\pm}_{\mathscr{A}}$ correspondences. If k field ext of F char 0, sign conjecture implies

$$C^+(Z) \Rightarrow C_{NC}(\mathscr{D}^{dg}_{perf}(Z))$$

i.e. on commutative motives more likely to hold than sign conjecture

• **Thm 3**: k and F char 0, one extension of other: if C_{NC} holds then change of symmetry isomorphism in tensor structure gives category $\mathrm{NNum}^{\dagger}_F(k)$ Tannakian

• **Thm 4**: standard conjecture $D_{NC}(\mathscr{A})$:

$$K_0(\mathscr{A})_F / \sim_{hom} = K_0(\mathscr{A})_F / \sim_{num}$$

homological defined by periodic cyclic homology: kernel of

$$K_0(\mathscr{A})_F = \mathrm{Hom}_{\mathrm{NChow}_F(k)}(k, \mathscr{A}) \xrightarrow{\overline{HP}_*} \mathrm{Hom}_{sVect(K)}(\overline{HP}_*(k), \overline{HP}_*(\mathscr{A}))$$

when k field ext of F char 0: $D(Z) \Rightarrow D_{NC}(\mathscr{D}^{dg}_{perf}(Z))$
i.e. for commutative motives more likely to hold than D conjecture

• **Thm 5**: F ext of k char 0: if C_{NC} and D_{NC} hold then $\mathrm{NNum}_F^{\dagger}(k)$ is a neutral Tannakian category with periodic cyclic homology as fiber functor

• **Thm 6**: k char 0: if C, D and C_{NC}, D_{NC} hold then

$$s\mathrm{Gal}(\mathrm{NNum}_k(k) \twoheadrightarrow \mathrm{Ker}(t : s\mathrm{Gal}(\mathrm{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$$

$$\mathrm{Gal}(\mathrm{NNum}_k^{\dagger}(k) \twoheadrightarrow \mathrm{Ker}(t : \mathrm{Gal}(\mathrm{Num}_k^{\dagger}(k)) \twoheadrightarrow \mathbb{G}_m)$$

where t induced by inclusion of Tate motives in the category of (commutative) numerical motives

(using periodic cyclic homology and de Rham cohomology)