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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Maria Chlouveraki

Talk Title: Symplectic reflection algebras I

Date: 01/31/13 **Time:** 3:45 am / pm (circle one)

List 6-12 key words for the talk: symplectic reflection group, symplectic resolution, conjugation invariant function, pseudo-reflections, spherical subalgebra

Please summarize the lecture in 5 or fewer sentances: Present some basic definitions and study how symplectic reflection algebras and their results allow us to determine the existence of symplectic resolutions. Discuss Etingof + Ginzburg's result on PBW property of a symplectic reflection algebra, its spherical subalgebra and double centralizer property.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - <u>**Computer Presentations**</u>: Obtain a copy of their presentation
  - <u>**Overhead**</u>: Obtain a copy or use the originals and scan them
  - <u>**Blackboard**</u>: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - <u>**Handouts**</u>: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to <u>notes@msri.org</u> with the workshop name and your name in the subject line.

01/31/13 Maria Chlouveraki: "Symplectic reflection algebras I"
3:45pm

- Symplectic reflection algebras were introduced by
  - Etingof + Ginzburg (2002)
  - Drinfeld (1986)
  - (dim 2) Crowley, Borey & Holland

- let V be a vector space over $\mathbb{C}$, $\dim_{\mathbb{C}} V = n < \infty$
  - $G \subset GL(V)$ finite
  - $\mathbb{C}[V]$ = set of regular functions on V = $Sym(V^*)$
  - $G \circlearrowright \mathbb{C}[V]$ with action:

$$^g f(v) = f(g^{-1}v), \quad \forall f \in \mathbb{C}[V], \ v \in V, g \in G$$

$$\mathbb{C}[V]^G = \{ f \in \mathbb{C}[V] \mid {}^g f = f, \ \forall g \in G \} \text{ invariants}$$

<u>Problem</u>. Understand the variety $V/G = \operatorname{Spec} \mathbb{C}[V]^G$

<u>Theorem</u>: (Shephard - Todd, Chevalley - Serre): TFAE:
1) V/G is smooth
2) $\mathbb{C}[V]^G$ is a polynomial algebra on n generators
3) G is a complex reflection group, i.e. a finite group generated by pseudo - reflections

<u>Defn</u>: A <u>pseudo-reflection</u> (or complex reflection) is an element $s \in GL(V)$ of finite order s.t. $\dim_{\mathbb{C}} \underbrace{\operatorname{Ker}(s - id_V)}_{H_s^*} = n-1$

$\Updownarrow$

$\operatorname{rank}(s - id_V) = 1$

$$S \sim \begin{pmatrix} z & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$ finite Coxeter group $\Rightarrow z = 1$

Classification of irred complex ref. group : (Shephard-Todd)

+ $G(\ell, p, n)$ such that $\frac{\ell}{p} \in \mathbb{Z}$
+ $G_4, G_5, \dots, G_{37}$

e.g. $G(\ell, 1, n) \cong (\mathbb{Z}/\ell\mathbb{Z}) \wr S_n \cong (\mathbb{Z}/\ell\mathbb{Z})^n \rtimes S_n$

$\ell = 1 \Rightarrow S_n$, $\ell = 2 \rightsquigarrow B_n$

$$\mathbb{C}[x_1, \dots, x_n]^{S_n} = \mathbb{C}[\Sigma_1, \dots, \Sigma_n]$$
$$\mathbb{C}[x_1, \dots, x_n]^{G(\ell,1,n)} = \mathbb{C}[F_1, \dots, F_n]$$
where $F_i = \Sigma_i(x_1^\ell, \dots, x_n^\ell)$

non e.g. $n = 2$, $V = \mathbb{C}^2$, $G \leq SL_2(\mathbb{C})$ finite, nontrivial

eg. $G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

G is not a complex ref. group

$\mathbb{C}^2/G$: Kleinian singularity

Defn.: The triple (G, V, ω) is a symplectic reflection group if

+ (V, ω) is a symplectic vector space
+ $G \leq Sp(V)$, ie. ω is invariant under G
+ G is generated by the set S = the set of all symplectic reflections in G $\rightsquigarrow rk(s - id_V) = 2$

Problems: Does V/G admit a symplectic resolution?

Defn.: $V = h \oplus h^*$ standard symplectic form

$$\omega_V(y_1 \oplus x_1, y_2 \oplus x_2) = x_2(y_1) - x_1(y_2)$$

eg. $V = \mathbb{C} \oplus \mathbb{C}^*$

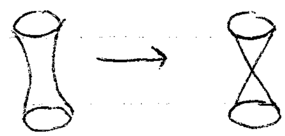
$G = \mathbb{Z}/2\mathbb{Z}$ acting on V by multiplication by -1

ω_V = standard symplectic form

$$\mathbb{C}[V]^G = \mathbb{C}[x^2, xy, y^2] = \mathbb{C}[A,B,C]/(AC-B^2)$$

$\mathbb{C}/G$ admits a symplectic resolution

$T^*\mathbb{P}^1$:



Rmk: V/G admits a symplectic resolution

$$\Downarrow \quad \not\Uparrow$$

(G, V, ω) is a symplectic reflection group

- Classification of symplectic reflection group:
(Haffman-Wales, Cohen, Guralnick-Saxl.)

+ $\Gamma \leq SL_2(\mathbb{C})$, $V = \underbrace{\mathbb{C}^2 \oplus \ldots \oplus \mathbb{C}^2}_{n \text{ times}}$

ω_V induced symplectic form from $\mathbb{C}^2$

$$G = \Gamma \wr S_n$$

$$\mathbb{C}^2/\Gamma \longleftarrow \mathbb{C}/\Gamma$$

$$\mathbb{C}^{2n}/\Gamma \wr S_n \longleftarrow \mathrm{Hilb}^n(\widehat{\mathbb{C}^2/\Gamma})$$

+ $G \leq GL(h)$ complex reflection group

$V = h \oplus h^*$, h/G smooth, V/G never smooth, sometimes there exists a symplectic resolution

ω_V = standard symplectic form

Symplectic reflections of $(G, V, \omega) \equiv$ pseudo-reflections of $(G, \mathfrak{h})$
↳ finite number of cases for dim $V \leq 10$

— From now on, (G, V, ω) is a symplectic reflection group

Defn: The skew group ring $\mathbb{C}[V] \rtimes G \cong \mathbb{C}[V] \otimes \mathbb{C}G$ as vector space such that $g \cdot f = {}^g f \cdot g$, $\forall f \in \mathbb{C}[V], g \in G$

Exercise: $\underset{\text{center}}{Z}(\mathbb{C}[V] \rtimes G) = \mathbb{C}[V]^G$

— let $s \in S$ $\qquad V^* = V$

$$\omega_s = \begin{cases} \omega & \text{on } \mathrm{Im}(s - id_V) \\ 0 & \text{on } \mathrm{Ker}(s - id_V) \end{cases} \qquad \omega_{V^*} = \omega$$

TV^* = tensor algebra $= k \oplus V^* \oplus (V^* \otimes V^*) \oplus \dots$

Defn: Symplectic reflection algebra

$$H_{t,\underline{c}} = TV^* \rtimes G \Big/ \left\langle [u,v] - t\omega(u,v) - 2\sum_{s \in S} \underbrace{\underline{c}(s)\omega_s(u,v)s}_{\mathbb{C}[G]} \,\Big|\, u, v \in V^* \right\rangle$$

$t \in \mathbb{C}$

$\underline{c}$ is a conjugation invariant function $S \to \mathbb{C}$

$\underline{c}(s) = \underline{c}(gsg^{-1})$, $\forall s \in S, g \in G$

Ex 1: $\mu_2 = \langle s \rangle$ acting on $\mathbb{C}^2$, $(\mathbb{C}^2)^* = \mathrm{Span}\langle x, y \rangle$

$s \cdot x = -x$, $s \cdot y = -y$, $\omega\langle y, x \rangle = 1$

$$H_{t,\underline{c}}(\mu_2) = \mathbb{C}\langle x, y, s \rangle \Big/ \left(\begin{array}{l} s^2 = 1 \\ sx = -xs \\ sy = -ys \\ [y, x] = t - 2\underline{c}(s) \end{array} \right)$$

Ex 2: $V = \mathbb{C}^2$, $SL_2(\mathbb{C}) \geq G$ symp. ref. group

Every $g \neq 1$ is a symplectic reflection and $\omega_g = \omega$

$(\mathbb{C}^2)^* = \mathrm{Span}\langle x, y\rangle \qquad \omega(x,y) = 1$

$$H_{t,\underline{c}}(G) = \mathbb{C}\langle x,y\rangle \rtimes G \Big/ [y,x] = t - 2\sum_{\substack{g\in G \\ g\neq 1}} \underline{c}(g)g$$

<u>Remarks</u>: 1) $\lambda \in \mathbb{C}^\times$ then

$$H_{\lambda t, \lambda \underline{c}} \cong H_{t,\underline{c}} \begin{cases} t=0 \\ t=1 \end{cases}$$

2) $H_{0,0}(G) = \mathbb{C}[V] \rtimes G$

<u>PBW</u>: (Etingof + Ginzburg)

Filtration F, $V^* \rightsquigarrow$ dim 1

$G \rightsquigarrow$ dim 0

$\mathrm{gr}(H_{t,\underline{c}}) \cong \mathbb{C}[V] \rtimes G$, as algebras

$H_{t,\underline{c}} \cong \mathbb{C}[V] \otimes G$, as vector spaces

$\Rightarrow \exists$ explicit basis for $H_{t,\underline{c}}$

<u>Corollary</u>: 1) $H_{t,\underline{c}}$ is a Noetherian ring

2) $H_{t,\underline{c}}$ has finite global dimension

$Z(\mathbb{C}[V] \rtimes G) = \mathbb{C}[V]^G$

Now $\mathbb{C}[V] \rtimes G$ contains another subalgebra isomorphic to $\mathbb{C}[V]^G$.

<u>Spherical subalgebra of $H_{t,\underline{c}}$</u>:

$$e := \frac{1}{|G|}\sum_{g\in G} g \in \mathbb{C}G \subseteq H_{t,\underline{c}}(G)$$

$$e^2 = e, \quad e(\mathbb{C}[V] \rtimes G)e \xrightarrow{\sim} \mathbb{C}[V]^G$$

$$efe \longleftarrow\!\shortmid f$$

let $\mathcal{U}_{t,\underline{c}} := e H_{t,\underline{c}} e$, this is the <u>spherical subalgebra</u>

PBW $\Rightarrow$ $\mathbb{C}[V]^G \cong e(\mathbb{C}[V] \rtimes G) \cong \operatorname{gr} \mathcal{U}_{t,\underline{c}}$

so $\mathbb{C}[V]^G \cong \mathcal{U}_{t,\underline{c}}$

filtration induced F

<u>Ex</u>: $G = \mu_2 = \langle s \rangle$, $e = \frac{1}{2}(1+s)$

$\mathcal{U}_{t,\underline{c}}(\mu_2)$ is generated by $\underline{h} = -\frac{1}{2} e(xy+yx)e$

$$\underline{e} = \frac{1}{2} e x^2 e$$

$$\underline{f} = \frac{1}{2} e y^2 e$$

$$[\underline{e}, \underline{f}] = t\underline{h}$$
$$[\underline{h}, \underline{e}] = -2t\underline{e}$$
$$[\underline{h}, \underline{f}] = 2t\underline{f}$$
$$\underline{e} \cdot \underline{f} = (2c - \underline{h}/2)(t/2 - c - \underline{h}/2)$$

If $t = 0$; $\mathcal{U}_{0,\underline{c}}$ is commutative

If $t \neq 0$, $\mathcal{U}_{t,\underline{c}}$ is a quotient of $\mathcal{U}\,so(2,\mathbb{C})$

<u>Double centralizer property</u>:

The right $\mathcal{U}_{t,\underline{c}}$-module $H_{t,\underline{c}}e$ is reflexive

$$\operatorname{End}_{H_{t,\underline{c}}}(H_{t,\underline{c}}e)^{op} \cong \mathcal{U}_{t,\underline{c}}$$

$$\operatorname{End}_{\mathcal{U}^{op}_{t,\underline{c}}}(H_{t,\underline{c}}e) \cong H_{t,\underline{c}}$$

<u>Theorem</u>: If $t \neq 0$, $Z(\mathcal{U}_{t,\underline{c}}) = \mathbb{C}$.

If $t = 0$, $\mathcal{U}_{t,\underline{c}}$ is commutative

Satake isomorphism: $Z(H_{t,\underline{c}}) \cong Z(\mathcal{U}_{t,\underline{c}})$

Corollary: If $t \neq 0$, then $Z(H_{t,\underline{c}}) = \mathbb{C}$
If $t = 0$, then $Z(H_{t,\underline{c}}) = \mathcal{U}_{t,\underline{c}}$

When $t=0$:
$X_{\underline{c}}(G) := \operatorname{Spec} \mathcal{U}_{0,\underline{c}} = \operatorname{Spec} Z(H_{0,\underline{c}})$
generalized Calogero - Moser space

$X_{\underline{c}}(G)$ is smooth $\Leftrightarrow$ $\dim L = |G|$, for all simple $H_{0,\underline{c}}$-module L

Theorem: (Ginzburg - Kaledin, Namikawa)
"$\Rightarrow$" "$\Leftarrow$"

V/G admits a symplectic resolution

$\Updownarrow$

$X_{\underline{c}}(G)$ is smooth for generic $\underline{c}$.

Next Lecture: Rational Cherednik algebras
$G = W \leq GL(h)$ complex ref. group
$V = h \oplus h^*$
ω_V = standard symplectic form
$(-,-) : h \times h^* \to \mathbb{C}$
For $s \in S$, H_s reflect. hyperplane of s

basis of $\operatorname{Im}(s - \operatorname{id}_V)|_{h^*, h}$ $\left\{ \begin{array}{l} \alpha_s \in h^* \text{ such that } \operatorname{Ker} \alpha_s = H_s \\ \alpha_s^\vee \in h \text{ such that } (\alpha_s^\vee, \alpha_V) = 1 - \det(s) \end{array} \right.$

$\underline{c} : S \to \mathbb{C}$ conj. inv. function
$(*)$: $[x_1, x_2] = 0$, $[y_1, y_2] = 0$

$(**)$: $[y,x] = t(y,x) - 2\sum_{s\in S} \frac{\underline{c}(s)}{1-\det(s)} (y\alpha_s)(\alpha_s^\vee, x)\, s$

for $x, x_1, x_2 \in \mathfrak{h}^*$

$y, y_1, y_2 \in \mathfrak{h}$

$H_{t,\underline{c}} = TV^* \rtimes W \Big/ \underset{(*),\,(**)}{\text{relations}}$

(continued lecture II on 02/01/13)