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17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen Email/Phone: van.nguyen3@gmail.com

Speaker's Name: Matilde Marcolli

Talk Title: Noncommutative motives and their applications II

Date: 02/01/13 Time: 11:00 am / pm (circle one)

List 6-12 key words for the talk: Tannakian category, noncommutative homological motives, motivic Galois groups, Tate triples, motivic decomposition

Please summarize the lecture in 5 or fewer sentences: Discuss in depth the proofs and applications of the main theorems / results.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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Thm 1: Schur finiteness $\overline{HH} : \mathrm{NChow}_F(k) \to \mathscr{D}_c(F)$
F-linear symmetric monoidal functor (Hochschild homology)

$$(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HH}))^\natural \to \mathscr{D}_c(F)$$

faithful F-linear symmetric monoidal

$\mathscr{D}_c(\mathscr{A})$ = full triang subcat of compact objects in $\mathscr{D}(\mathscr{A}) \Rightarrow \mathscr{D}_c(F)$ identified with fin-dim $\mathbb{Z}$-graded F-vector spaces: Shur finite

general fact: $L : \mathscr{C}_1 \to \mathscr{C}_2$ F-linear symmetric monoidal functor: $X \in \mathscr{C}_1$ Schur finite $\Rightarrow L(X) \in \mathscr{C}_2$ Schur finite; L faithful then also converse: $L(X) \in \mathscr{C}_2$ Schur finite $\Rightarrow X \in \mathscr{C}_1$ Schur finite

conclusion: $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HH}))^\natural$ is Schur finite

also $\mathrm{Ker}(\overline{HH}) \subset \mathscr{N}$ with F-linear symmetric monoidal functor $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HH}))^\natural \to (\mathrm{NChow}_F(k)/\mathscr{N})^\natural = \mathrm{NNum}_F(k)$

$\Rightarrow \mathrm{NNum}_F(k)$ Schur finite $\Rightarrow$ super-Tannakian

Thm 2: periodic cyclic homology

mixed complex (M, b, B) with $b^2 = B^2 = Bb + bB = 0$, $\deg(b) = 1 = -\deg(B)$: *periodized*

$$\cdots \prod_{n\ even} M_n \stackrel{b+B}{\to} \prod_{n\ odd} M_n \stackrel{b+B}{\to} \prod_{n\ even} M_n \cdots$$

periodic cyclic homology (the derived cat of $\mathbb{Z}/2\mathbb{Z}$-graded complexes

$$HP : \mathtt{dgcat} \to \mathscr{D}_{\mathbb{Z}/2\mathbb{Z}}(k)$$

induces F-linear symmetric monoidal functor

$$\overline{HP}_* : \mathrm{NChow}_F(k) \to s\mathrm{Vect}(F)$$

or to $s\mathrm{Vect}(k)$ if k field ext of F

Note the issue here:

• mixed complex functor symmetric monoidal but 2-periodization not (infinite product don't commute with $\otimes$)

• *lax symmetric monoidal* with $\mathscr{D}_{\mathbb{Z}/2\mathbb{Z}}(k) \simeq S\mathrm{Vect}(k)$ (not fin dim)

• $HP : \mathtt{dgcat} \to S\mathrm{Vect}(k)$ *additive invariant*: through $\mathtt{Hmo}_0(k)$

• $\mathrm{NChow}_F(k) = (\mathtt{Hmo}_0(k)^{sp})^\natural_F$ (sp = gen by smooth proper dgcats)

• periodic cyclic hom *finite dimensional* for smooth proper dgcats + a result of Emmanouil

$\Rightarrow$ lax symmetric monoidal $\overline{HP}_* : \mathtt{Hmo}_0(k)^{sp} \to s\mathrm{Vect}(k)$ is symmetric monoidal

standard conjecture C_{NC} (Künneth type)

- $C_{NC}(\mathscr{A})$: Künneth projections

$$\pi^{\pm}_{\mathscr{A}} : \overline{HP}_*(\mathscr{A}) \twoheadrightarrow \overline{HP}^{\pm}_*(\mathscr{A}) \hookrightarrow \overline{HP}_*(\mathscr{A})$$

are algebraic: $\pi^{\pm}_{\mathscr{A}} = \overline{HP}_*(\underline{\pi}^{\pm}_{\mathscr{A}})$ image of correspondences

- then from Keller + Hochschild-Konstant-Rosenberg have

$\overline{HP}_*(\mathscr{D}^{dg}_{perf}(Z)) = HP_*(\mathscr{D}^{dg}_{perf}(Z)) = HP_*(Z) = \oplus_{n\, even/odd} H^n_{dR}(Z)$

- hence $C^+(Z) \Rightarrow C_{NC}(\mathscr{D}^{dg}_{perf}(Z))$ with $\underline{\pi}^{\pm}_{\mathscr{D}^{dg}_{perf}(Z)}$ image of $\underline{\pi}^{\pm}_Z$ under

$\mathrm{Chow}(k) \to \mathrm{Chow}(k)/_{-\otimes\mathbb{Q}(1)} \hookrightarrow \mathrm{NChow}(k)$

classical: (using deRham as Weil cohomology) $C(Z)$ for Z correspondence, the Künneth projections $\pi^n_Z : H^*_{dR}(Z) \twoheadrightarrow H^n_{dR}(Z)$ are algebraic, $\pi^n_Z = H^*_{dR}(\underline{\pi}^n_Z)$, with $\underline{\pi}^n_Z$ correspondences

sign conjecture: $C^+(Z)$: Künneth projectors $\pi^+_Z = \sum_{n=0}^{\dim Z} \pi^{2n}_Z$ are algebraic, $\pi^+_Z = H^*_{dR}(\underline{\pi}^+_Z)$ (hence π^-_Z also)

Thm 3: Tannakian category first steps

• have F-linear symmetric monoidal and also full and essentially surjective functor: $\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*) \to \mathrm{NChow}_F(k)/\mathscr{N}$

• assuming $C_{NC}(\mathscr{A})$: have $\underline{\pi}^{\pm}_{(\mathscr{A},e)} = e \circ \underline{\pi}^{\pm}_{\mathscr{A}} \circ e$; if $\underline{X}$ trivial in $\mathrm{NChow}_F(k)/\mathscr{N}$ intersection numbers $\langle \underline{X}^n, \underline{\pi}^{\pm}_{(\mathscr{A},e)} \rangle$ vanishes ($\mathscr{N}$ is $\otimes$-ideal)

• intersection number is categorical trace of $\underline{X}^n \circ \underline{\pi}^{\pm}_{(\mathscr{A},e)}$ (M.M., G.Tabuada, 1105.2950)

$$\Rightarrow \mathrm{Tr}(\overline{HP}_*(\underline{X}^n \circ \underline{\pi}^{\pm}_{(\mathscr{A},e)}) = \mathrm{Tr}(\overline{HP}^{\pm}_*(\underline{X})^n) = 0$$

trace all n-compositions vanish $\Rightarrow$ nilpotent $\overline{HP}^{\pm}_*(\underline{X})$

• conclude: nilpotent ideal as kernel of

$$\mathrm{End}_{\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*)}(\mathscr{A}, e) \twoheadrightarrow \mathrm{End}_{\mathrm{NChow}_F(k)/\mathscr{N}}(\mathscr{A}, e)$$

• then functor $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*))^\natural \to \mathrm{NNum}_F(k)$ full conservative essentially surjective: (quotient by $\mathscr{N}$ full and ess surj; idempotents can be lifted along surj F-linear homom with nilpotent

Tannakian category: modification of tensor structure

- $H : \mathscr{C} \to s\mathrm{Vect}(K)$ symmetric monoidal F-linear (K ext of F) faithful, Künneth projectors $\pi_N^{\pm} = H(\underline{\pi}_N^{\pm})$ for $\underline{\pi}_N^{\pm} \in \mathrm{End}_{\mathscr{C}}(N)$ for all $N \in \mathscr{C}$ then modify symmetry isomorphism

$$c^{\dagger}_{N_1,N_2} = c_{N_1,N_2} \circ (e_{N_1} \otimes e_{N_2}) \quad \text{with } e_N = 2\underline{\pi}_N^{+} - id_N$$

- get F-linear symmetric monoidal functor $\mathscr{C}^{\dagger} \xrightarrow{H} s\mathrm{Vect}(K) \to \mathrm{Vect}(K)$

- if $P : \mathscr{C} \to \mathscr{D}$, F-linear symmetric monoidal (essentially) surjective, then $P : \mathscr{C}^{\dagger} \to \mathscr{D}^{\dagger}$ (use image of e_N to modify $\mathscr{D}$ compatibly)

- apply to functors $\overline{HP}_* : (\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*))^{\natural} \to s\mathrm{Vect}(K)$ and $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*))^{\natural} \to \mathrm{NNum}_F(k)$

$\Rightarrow$ obtain $\mathrm{NNum}^{\dagger}_F(k)$ satisfying Deligne's intrinsic characterization for Tannakian: with $\tilde{N}$ lift to $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*))^{\natural,\dagger}$ have

$$\mathrm{rk}(N) = \mathrm{rk}(\overline{HP}_*(\tilde{N})) = \dim(\overline{HP}^{+}_*(\tilde{N})) + \dim(\overline{HP}^{-}_*(\tilde{N})) \geq 0$$

Thm 4: Noncommutative homological motives

$$\overline{HP}_* : \mathrm{NChow}_F(k) \to s\mathrm{Vect}(K)$$

$$K_0(\mathscr{A})_F = \mathrm{Hom}_{\mathrm{NChow}_F(k)}(k, \mathscr{A}) \stackrel{\overline{HP}_*}{\to} \mathrm{Hom}_{s\mathrm{Vect}(K)}(\overline{HP}_*(k), \overline{HP}_*(\mathscr{A}))$$

kernel gives homological equivalence $K_0(\mathscr{A})_F$ mod $\sim_{hom}$

- $D_{NC}(\mathscr{A})$ standard conjecture:

$$K_0(\mathscr{A})_F / \sim_{hom} = K_0(\mathscr{A})_F / \sim_{num}$$

- on $\mathrm{Chow}_F(k)/_{-\otimes\mathbb{Q}(1)}$ induces homological equivalence with sH_{dR} (de Rham even/odd) $\Rightarrow \mathscr{Z}^*_{hom}(Z)_F \twoheadrightarrow K_0(\mathscr{D}^{dg}_{perf}(Z))_F / \sim_{hom}$
- classical cycles $\mathscr{Z}^*_{hom}(Z)_F \simeq \mathscr{Z}^*_{num}(Z)_F$; for numerical $\mathscr{Z}^*_{num}(Z)_F \stackrel{\sim}{\to} K_0(\mathscr{D}^{dg}_{perf}(Z))_F / \sim_{num}$; then get

$$D(Z) \Rightarrow D_{NC}(\mathscr{D}^{dg}_{perf}(Z))$$

Thm 5: assume C_{NC} and D_{NC} then

$$\overline{HP}_* : \mathrm{NNum}^{\dagger}_F(k) \to \mathrm{Vect}(F)$$

exact faithful $\otimes$-functor: fiber functor $\Rightarrow$ *neutral* Tannakian category $\mathrm{NNum}^{\dagger}_F(k)$

Thm 6: Motivic Galois groups

• Galois group of neutral Tannakian category $\mathrm{Gal}(\mathrm{NNum}^{\dagger}_F(k))$ want to compare with commutative case $\mathrm{Gal}(\mathrm{Num}^{\dagger}_F(k))$

• super-Galois group of super-Tannakian category $s\mathrm{Gal}(\mathrm{NNum}_F(k))$ compare with commutative motives case $s\mathrm{Gal}(\mathrm{Num}_F(k))$

• related question: what are truly noncommutative motives?

Tate triples (Deligne–Milne)

- For $A = \mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{G}_m$ or μ_2, Tannakian cat $\mathscr{C}$ with A-grading: A-grading on objects with $(X \otimes Y)^a = \oplus_{a=b+c} X^b \otimes Y^c$; homom $w : B \to \underline{\mathrm{Aut}}^{\otimes}(id_{\mathscr{C}})$ (weight); central hom $B \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$

- Tate triple $(\mathscr{C}, w, T)$: $\mathbb{Z}$-graded Tannakian $\mathscr{C}$ with weight w, invertible object T (Tate object) weight -2

- Tate triple $\Rightarrow$ central homom $w : \mathbb{G}_m \to \mathrm{Gal}(\mathscr{C})$ and homom $t : \mathrm{Gal}(\mathscr{C}) \to \mathbb{G}_m$ with $t \circ w = -2$.

- $H = Ker(t : \mathrm{Gal}(\mathscr{C}) \to \mathbb{G}_m)$ defines Tannakian category $\simeq \mathrm{Rep}(H)$. It is the “quotient Tannakian category" (Milne) of inclusion of subcategory gen by Tate object into $\mathscr{C}$

Galois group and orbit category

• $\mathscr{T} = (\mathscr{C}, w, T)$ Tate triple, $\mathscr{S} \subset \mathscr{C}$ gen by T, pseudo-ab envelope $(\mathscr{C}/_{-\otimes T})^\natural$ of orbit cat $\mathscr{C}/_{-\otimes T}$ is neutral Tannakian with

$$\mathrm{Gal}((\mathscr{C}/_{-\otimes T})^\natural) \simeq \mathrm{Ker}(t : \mathrm{Gal}(\mathscr{C}) \twoheadrightarrow \mathbb{G}_m)$$

• Quotient Tannakian categories with resp to a fiber functor (Milne): $\omega_0 : \mathscr{S} \to \mathrm{Vect}(F)$ then $\mathscr{C}/\omega_0$ pseudo-ab envelope of $\mathscr{C}'$ with same objects as $\mathscr{C}$ and morphisms $\mathrm{Hom}_{\mathscr{C}'}(X, Y) = \omega_0(\underline{Hom}_{\mathscr{C}}(X, Y)^H)$ with X^H largest subobject where H acts trivially

• fiber functor $\omega_0 : X \mapsto \mathrm{colim}_n \mathrm{Hom}_{\mathscr{C}}(\oplus_{r=-n}^n \mathbf{1}(r), X) \in \mathrm{Vect}(F)$
$\Rightarrow$ get $\mathscr{C}' = \mathscr{C}/_{-\otimes T}$

super-Tannakian case: super Tate triples

• Need a super-Tannakian version of Tate triples

• super Tate triple: $\mathscr{S}\mathscr{T} = (\mathscr{C}, \omega, \underline{\pi}^{\pm}, \mathscr{T}^{\dagger})$ with $\mathscr{C}$ = neutral super-Tannakian; $\omega : \mathscr{C} \to s\mathrm{Vect}(F)$ super-fiber functor; idempotent endos: $\omega(\underline{\pi}_X^{\pm}) = \pi_X^{\pm}$ Künneth proj.; neutral Tate triple $\mathscr{T}^{\dagger} = (\mathscr{C}^{\dagger}, w, T)$ with $\mathscr{C}^{\dagger}$ modified symmetry constraint from $\mathscr{C}$ using $\underline{\pi}^{\pm}$

• assuming C and D: a super Tate triple for (comm) num motives

$$(\mathrm{Num}_k(k), \overline{sH}^*_{dR}, \underline{\pi}_X^{\pm}, (\mathrm{Num}_k^{\dagger}(k), w, \mathbb{Q}(1)))$$

super-Tannakian case: orbit category

- $\mathscr{S}\mathscr{T} = (\mathscr{C}, \omega, \underline{\pi}^{\pm}, \mathscr{T}^{\dagger})$ super Tate triple; $\mathscr{S} \subset \mathscr{C}$ full neutral super-Tannakian subcat gen by T

- Assume: $\underline{\pi}^{-}_{T}(T) = 0$; for $K = \mathrm{Ker}(t : \mathrm{Gal}(\mathscr{C}^{\dagger}) \to \mathbb{G}_m)$ of Tate triple $\mathscr{T}^{\dagger}$, if $\epsilon : \mu_2 \to H$ induced $\mathbb{Z}/2\mathbb{Z}$ grading from $t \circ w = -2$; then (H, ϵ) super-affine group scheme is Ker of $s\mathrm{Gal}(\mathscr{C}) \to s\mathrm{Gal}(\mathscr{S})$ and $\mathrm{Rep}_F(H, \epsilon) = \mathrm{Rep}^{\dagger}_F(H)$.

- Conclusion: pseudoabelian envelope of $\mathscr{C}/_{-\otimes T}$ is neutral super-Tannakian and seq of exact $\otimes$-functors $\mathscr{S} \subset \mathscr{C} \to (\mathscr{C}/_{-\otimes T})^{\natural}$ gives

$$s\mathrm{Gal}((\mathscr{C}/_{-\otimes T})^{\natural}) \xrightarrow{\sim} \mathrm{Ker}(t : s\mathrm{Gal}(\mathscr{C}) \to \mathbb{G}_m)$$

- have also $(\mathscr{C}^{\dagger}/_{-\otimes T})^{\natural} \simeq (\mathscr{C}/_{-\otimes T})^{\natural,\dagger} \simeq \mathrm{Rep}^{\dagger}_F(H, \epsilon) \simeq \mathrm{Rep}_F(H)$

Then for Galois groups:

• then surjective $\mathrm{Gal}(\mathrm{NNum}_k^\dagger(k)) \twoheadrightarrow \mathrm{Gal}((\mathrm{Num}_k^\dagger(k)/_{-\otimes\mathbb{Q}(1)})^\natural)$ from embedding of subcategory and
$\mathrm{Gal}((\mathrm{Num}_k^\dagger(k)/_{-\otimes\mathbb{Q}(1)})^\natural) = \mathrm{Ker}(t : \mathrm{Num}_k^\dagger(k) \to \mathbb{G}_m)$

• for super-Tannakian: surjective (from subcategory)
$s\mathrm{Gal}(\mathrm{NNum}_k(k)) \twoheadrightarrow s\mathrm{Gal}((\mathrm{Num}_k(k)/_{-\otimes\mathbb{Q}(1)})^\natural)$ and
$s\mathrm{Gal}((\mathrm{Num}_k(k)/_{-\otimes\mathbb{Q}(1)})^\natural) \simeq \mathrm{Ker}(t : s\mathrm{Gal}(\mathrm{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$

• What is kernel? Ker = “truly noncommutative motives"

$$\mathrm{Gal}(\mathrm{NNum}_k^\dagger(k)) \twoheadrightarrow \mathrm{Ker}(t : \mathrm{Num}_k^\dagger(k) \to \mathbb{G}_m)$$

$$s\mathrm{Gal}(\mathrm{NNum}_k(k)) \twoheadrightarrow \mathrm{Ker}(t : s\mathrm{Gal}(\mathrm{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$$

what do they look line? examples? general properties?

Using NC motives to study commutative motives

Example: full exceptional collections and motivic decompositions

Examples of motivic decompositions:

- Projective spaces: $h(\mathbb{P}^n) = 1 \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^n$
- Quadrics (k alg closed char 0):

$$h(Q_q)_{\mathbb{Q}} \simeq \begin{cases} 1 \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^{\otimes n} & d \text{ odd} \\ 1 \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^{\otimes n} \oplus \mathbb{L}^{\otimes (d/2)} & d \text{ even}. \end{cases}$$

- Fano 3-folds:

$$h(X)_{\mathbb{Q}} \simeq 1 \oplus h^1(X) \oplus \mathbb{L}^{\oplus b} \oplus (h^1(J) \otimes \mathbb{L}) \oplus (\mathbb{L}^{\otimes 2})^{\oplus b} \oplus h^5(X) \oplus \mathbb{L}^{\otimes 3},$$

$h^1(X)$ and $h^5(X)$ Picard and Albanese motives, $b = b_2(X) = b_4(X)$
J abelian variety (isogenous to intermediate Jacobian if $k = \mathbb{C}$)

Full exceptional collections in the derived category $\mathscr{D}^b(X)$

A collection of objects $\{E_1, \ldots, E_n\}$ in a F-linear triangulated category $\mathscr{C}$ is *exceptional* if $\mathrm{RHom}(E_i, E_i) = F$ for all i and $\mathrm{RHom}(E_i, E_j) = 0$ for all $i > j$; it is *full* if $\mathscr{C}$ is minimal triangulated subcategory containing it.

Examples of full exceptional collections:

- Projective spaces (Beilinson): $(\mathscr{O}(-n), \ldots, \mathscr{O}(0))$
- Quadrics (Kapranov):

$$\begin{array}{rl} (\Sigma(-d), \mathscr{O}(-d+1), \ldots, \mathscr{O}(-1), \mathscr{O}) & \text{if } d \text{ is odd} \\ (\Sigma_+(-d), \Sigma_-(-d), \mathscr{O}(-d+1), \ldots, \mathscr{O}(-1), \mathscr{O}) & \text{if } d \text{ is even}\,, \end{array}$$

$\Sigma_\pm$ (and Σ) spinor bundles

• Toric varieties (Kawamata)

• Homogeneous space (Kuznetsov-Polishchuk)
Conjecture (KP): k alg cl char 0, parabolic subgroup $P \subset G$ of semisimple alg group then $\mathscr{D}^b(G/P)$ has full exceptional collection

• Fano 3-folds with vanishing odd cohomology (Ciolli)

• Moduli spaces of rational curves $\overline{\mathscr{M}}_{0,n}$ (Manin–Smirnov)

Note: all these cases also have motivic decompositions

Reason: exceptional collections and motivic decompositions are related through the relation between commutative and NC motives

Thm 7: Full exceptional collections and motivic decompositions

if $\mathscr{D}^b(X)$ has a full exceptional collection, then $h(X)_{\mathbb{Q}}$ has a motivic decomposition

$$h(X)_{\mathbb{Q}} \simeq \mathbb{L}^{\ell_1} \oplus \cdots \oplus \mathbb{L}^{\ell_m}$$

for some $\ell_1, \ldots, \ell_m \geq 0$

Note: works also for Deligne–Mumford stacks

- $\mathscr{D}^b_{dg}(X)$ unique dg enhancement: $\langle E_j \rangle_{dg} \simeq \mathscr{D}^b_{dg}(k)$
- Look at corresponding elements in $\mathrm{NChow}_{\mathbb{Q}}(k)$ under universal localizing invariant $\mathscr{U} : \mathtt{dgcat}(k) \to \mathrm{NChow}_{\mathbb{Q}}(k)$

$$\oplus_{j=1}^m \mathscr{U}(\mathscr{D}^b_{dg}(k)) \stackrel{\simeq}{\to} \mathscr{U}(\mathscr{D}^b_{dg}(X))$$

from inclusions of dg categories $\langle E_j \rangle_{dg} \hookrightarrow \mathscr{D}^b_{dg}(X)$

using (Tabuada “Higher K-theory via universal invariants"): given split short exact sequence of pre-triangulated dg categories

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{B} \arrow[r, "\iota_{\mathscr{B}}"'] & \mathscr{A} \arrow[l, bend right] \arrow[r] & \mathscr{C} \arrow[l, bend right, "\iota_{\mathscr{C}}"'] \arrow[r] & 0 \end{tikzcd}$$

mapped by universal localizing invariant $\mathscr{U}(-)$ to a distinguished split triangle so $\mathscr{U}(\mathscr{B}) \oplus \mathscr{U}(\mathscr{C}) \stackrel{\sim}{\to} \mathscr{U}(\mathscr{A})$
Applied to

$$\mathscr{A} := \langle E_i, \cdots, E_m \rangle_{dg}, \quad \mathscr{B} := \langle E_i \rangle_{dg}, \quad \mathscr{C} := \langle E_{i+1}, \ldots, E_m \rangle_{dg}$$

gives

$$\mathscr{U}(\mathscr{D}^b_{dg}(k)) \oplus \mathscr{U}(\langle E_{i+1}, \ldots, E_m \rangle_{dg}) \stackrel{\sim}{\to} \mathscr{U}(\langle E_i, \ldots, E_m \rangle_{dg})$$

recursively get result using $\mathscr{D}^b_{dg}(X) = \langle E_1, \ldots, E_m \rangle_{dg}$

A consequence: Hodge–Tate cohomology

Thm 8: If a smooth complex projective variety V has a full exceptional collection then it is Hodge–Tate (Hodge numbers $h^{p,q}(V) = 0$ for $p \neq q$)

Reason: motivic decomposition

Dubrovin conjecture: V smooth projective complex

(i) Quantum cohomology of V is (generically) semi-simple if and only if V is Hodge-Tate and $\mathscr{D}^b(V)$ has a full exceptional collection.

(ii) Stokes matrix of structure connection of quantum cohomology = Gram matrix of exceptional collection

$$\chi : K_0(V) \times K_0(V) \to \mathbb{Z}, \quad \sum_{n \in \mathbb{Z}} (-1)^n \dim \mathrm{Ext}^n(\mathscr{F}_1, \mathscr{F}_2)$$

First observation: Hodge-Tate hypothesis not necessary

Quantum cohomology and motives

Question: is QH^* motivic? (Manin, ECM 2000)

• V smooth complex projective variety; Moduli space $\overline{\mathscr{M}_n(V,\beta)}$ of stable maps $(f : C \to V, x_1, \ldots, x_n, [C] = \beta \in H_2(V))$; map to $\bar{\mathscr{M}}_{0,n}$ to marked curve $(C, x_1, \ldots, x_n)$; map to V^n evaluation

• Behrend–Manin: Gromov–Witten invariants of genus zero give a correspondence in the category $\mathrm{DMChow}_{\mathbb{Q}}(k)/_{-\otimes\mathbb{Q}(1)}$

$$\phi_*(J_{V,\beta,n}) \subset \bar{\mathscr{M}}_{0,n} \times V^n$$

image of virtual fundamental class (Behrend–Fantechi) of $\overline{\mathscr{M}_n(V,\beta)}$

• cycles $\gamma \subset V^n$ pulled back to $\bar{\mathscr{M}}_{0,n} \times V^n$; intersect with $\phi_*(J_{V,\beta,n})$; push forward to $\bar{\mathscr{M}}_{0,n}$; project onto $\mathbb{L}^{n-3}$ component of motivic decomposition of $\bar{\mathscr{M}}_{0,n}$ $\Rightarrow$ *Gromov–Witten invariants* of genus zero

$$\gamma \mapsto \langle\gamma\rangle_{0,\beta,n}\mathbb{L}^{n-3}$$

• Gromov–Witten invariants are used to deform the intersection product in the ring structure of cohomology: quantum corrections through generating function (potential)

$$\begin{aligned}\Phi(x) = & \ \tfrac{1}{6}(\textstyle\sum_a x_a\gamma_a)^3 \\ + & \ \textstyle\sum_{\beta\neq 0} e^{\langle\beta,\sum_{\deg(\gamma_b)=2} x_b\gamma_b\rangle} q^\beta \sum_{\deg(\gamma_{a_i})\neq 2} \langle\gamma_{a_n}\cdots\gamma_{a_1}\rangle_{0,n,\beta} \frac{x_{a_1}\cdots x_{a_n}}{n!}\end{aligned}$$

• Frobenius manifold structure on $H^*(V,\mathbb{C})$: associative multiplication

$$\partial_a \circ \partial_b = \sum_c A_{ab}{}^c \partial_c, \quad A_{abc} = \partial_a\partial_b\partial_c\Phi$$

(associativity WDVV nonlinear differential equations for Φ)

• Quantum cohomology $QH^*(V)$ with new multiplication; *semisimple*: there is a basis in which the tensor A is diagonal.

Question: Can NC motives say more about Dubrovin conjecture?

Currently work in progress! (with Yuri Manin and Goncalo Tabuada)

Main ingredients:

• a motivic approach to Gromov–Witten invariants and Quantum Cohology (Behrend–Manin)

• recasting the GW correspondence of Behrend–Manin as a correspondence in NChow

• write its coefficients on a basis given by the exceptional collection

• constraints on the coefficients, from the exceptional collection

Hopefully, will report on progress later in the term!

Some bibliography:

• M.M., G. Tabuada, *Noncommutative motives, numerical equivalence, and semi-simplicity*, arXiv:1105.2950, to appear in American J. Math.

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• M.M., G. Tabuada, *Noncommutative numerical motives, Tannakian structures, and motivic Galois groups*, arXiv:1110.2438

• M.M., G. Tabuada, *Unconditional motivic Galois groups and Voevodsky's nilpotence conjecture in the noncommutative world*, arXiv:1112.5422

• M.M., G. Tabuada, *From exceptional collections to motivic decompositions via noncommutative motives*, arXiv:1202.6297

• M.M., G. Tabuada, *Noncommutative Artin motives*, arXiv:1205.1732

• M.M., G. Tabuada, *Jacobians of noncommutative motives*, arXiv:1212.1118