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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Van C. Nguyen **Email/Phone:** van.nguyen3@gmail.com

Speaker's Name: Graham Leuschke

Talk Title: Non-commutative desingularizations and MCM modules II

Date: 02/01/13 **Time:** 2:00 am / pm (circle one)

List 6-12 key words for the talk: Noncomm. crepant resolution, tilting bundle, derived equivalent, generic/symmetric/alternating matrices

Please summarize the lecture in 5 or fewer sentances: Discuss result on noncomm. crepant resolutions (NCCR) by Van den Bergh, Hailong Dao, Iyama, ect. followed by examples on $\mathbb{C}^*$-invariants, determinantal rings.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - <u>Computer Presentations</u>: Obtain a copy of their presentation
  - <u>Overhead</u>: Obtain a copy or use the originals and scan them
  - <u>Blackboard</u>: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - <u>Handouts</u>: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to <u>notes@msri.org</u> with the workshop name and your name in the subject line.

02/01/13
2:00pm

Graham Leuschke :
" Non-commutative desingularizations and MCM modules II "

Recall:

A non-commutative crepant resolution (NCCR) of a Gorenstein normal domain R is an R-algebra $\Lambda = \mathrm{End}_R(M)$ for a reflexive R-module M, which is MCM as R-module and of finite global dim.

Theorem. [Van den Bergh] If R is a 3-dim'l Gorenstein normal domain, $X = \mathrm{Spec}(R)$ and $\pi : Y \to X$ is a crepant resolution of singularities which has nice properties on fibers. Then R has a NCCR, and all crepant resolutions (C and NC) are derived equivalent.

Rmk: The final statement gives a second proof of a result of T. Bridgeland, verifying (in this special case) conjecture of Bondal-Orlov:

Conjecture: All geometric crepant resolutions of a variety X are derived equivalent.

One can also ask that all crepant resolutions - non comm. and comm. - be derived equivalent.

Iyama - Reiten, Iyama - Wemyss: All the NCCRs are derived equivalent in dim 3.

+ A word on the proof of Van den Bergh's result:

From $\Lambda = \mathrm{End}_R(M)$ to Y: moduli space of representations

From Y to $\Lambda = \mathrm{End}_R(M)$: cook up a <u>tilting</u> sheaf on Y, $\mathcal{M}$, and push it down.

It is <u>not</u> the case that existence of CCRs is equivalent to that of NCCRs in dim ≥ 4. There are obstructions.

- [Hailong Dao '10] : Let $R = S/(f)$ be a local hypersurface ring. For concreteness, let $S = \mathbb{C}[[x_0, \dots, x_n]]$
  Assume R satisfies :
  i) If R is $\mathbb{Q}$-factorial and $n = 3$, then R has no NCCR.
  ii) If R has an isolated singularity and $n \geq 4$ is <u>even</u>, then R has no NCCR.

- On the other hand, it's known [H.W. Lin] that
  $\mathbb{C}[[x_0, \dots, x_n]] / (x_0^r + x_1^n + \dots + x_n^n)$ has a CCR $\Leftrightarrow r \equiv 0$ or $1 \mod n$
  ↳ This give counterexample to "$\exists$ CCR $\Rightarrow \exists$ NCCR" statement.

- Also, $\mathbb{C}[[x, y, z, t]]^{\mathbb{Z}_2}$, $x \mapsto -x$, etc., has no CCR [Reid] but does have a NCCR by the McKay correspondence from last talk

- <u>Example</u>: k^*-invariants [Van den Bergh's]
  Let V be a vector space over a field k, basis $x_1, \dots, x_n$
  k^* acting on V by $\alpha \cdot x_i = \alpha^{a_i} x_i$, for some fixed $a_i \in \mathbb{Z}$
  Assume $\gcd(a_i, a_j) = 1$, $\forall i, j \in \{1, \dots, n\}$,
  at least 2 of the a_i's are positive and at least 2 are negative
  $S := k[V]$, $R = S^{k^*}$, invariants

If we let x_i have degree a_i, then S is $\mathbb{Z}$-graded, $R = S_0$, and each S_i is a f.g. R-module.

Known: $\mathrm{Hom}_R(S_i, S_j) = S_{j-i}$

[Stanley]: S_j is a MCM R-module $\iff \sum_{a_i<0} a_i < j < \sum_{a_i>0} a_i$

Assume $\sum_{i=0}^{n} a_i = 0$ (ie. R is Gorenstein)
then

$$\mathrm{End}_R(S_0 \oplus S_1 \oplus \dots \oplus S_N), \text{ where } N = \sum_{a_i>0} a_i$$

is a NCCR of R.

In this case,
MCMness satisfies. ✓ (as R-mod, its direct summands are exactly $S_{-N}, \dots, S_0, \dots, S_N$)
finite gl. dim is harder.

+ <u>Specific case</u>: let $S = k[a,b,c,d]$ be graded by assigning deg. deg. $1, 1, -1, -1$ respectively

Then $R = S_0 = k[ac, ad, bc, bd]$

$$\cong k[x,y,u,v]/(xv - yu)$$

is the cone over $\mathbb{P}^1 \times \mathbb{P}^1$

from Paul Smith's talk. let $I = S_1 = (a,b)R \cong (ac, bc) = (x,u)$

Then $\Lambda = \mathrm{End}_R(R \oplus I) = \begin{pmatrix} R & I^* \\ I & R \end{pmatrix}$
is a NCCR.

where $I^* = \mathrm{Hom}_R(I, R) \cong (x, v)$

↳ This connects with Paul's ring B in his talk:

$$B = k\langle r, s\rangle / \langle r^2s - sr^2, s^2r - rs^2\rangle$$

$R = B^{(2)}$ ← second Veronese

$$\begin{pmatrix} x = r^2 & u = sr \\ y = rs & v = s^2 \end{pmatrix}$$

Λ : $r = \begin{pmatrix} & 1/x \\ x & \end{pmatrix}$

$s = \begin{pmatrix} & 1/u \\ u & \end{pmatrix}$

We have: $R \subseteq B \subseteq \Lambda$

In fact, $\Lambda = B \rtimes \mathbb{Z}_2$, $r \leftrightarrow s$

<u>Examples</u>: Determinantal rings

Set up: let k be a field of char. 0

$X = (x_{ij})$ $\left\{\begin{array}{l} \text{generic} \\ \text{generic alternating} \\ \text{generic symmetric} \end{array}\right\}$ $n \times n$ matrix

$S = k[X] = k[\{x_{ij}\}]$

$R = S/I_{m+1}(X)$ ⟵ $(m+1) \times (m+1)$ minors / pfaffias

(When X is generic / generic alternating, R is Gorenstein)

View X as a homomorphism between free S-modules:

$$S^n \xrightarrow{X} S^n$$

and for a partition λ, define an R-module

$$M_\lambda = \text{Image}(S_\lambda(X^T)) \otimes_S R$$

Schur functor — transpose

and set

$$M := \bigoplus_\lambda M_\lambda$$, where λ are partitions with at <u>most</u> m rows and at most $n-m$ columns

M is a MCM R-module

<u>eg</u>. $m = n-1$, then $\lambda = \left.\boxed{\begin{array}{c} \\ \vdots \\ \\ \end{array}}\right\} i$ and $M_\lambda = \text{image}\, \Lambda^i(X^T) \otimes_S R$

<u>Theorem</u>: [Buchweitz - L. - Van den Bergh, Weyman - Zhao]

i) If X is generic, then $\text{End}_R(M)$ gives a NCCR.

ii) If X is symmetric, then $\mathrm{End}_R(M)$ has finite global dim., and is MCM $\iff m = n-1$

iii) If X is alternating, $\mathrm{End}_R(M)$ has finite global dim. and is <u>never</u> MCM.

In each case, the proof relies on realizing M and $\mathrm{End}_R(M) \overset{\text{def}}{=} \Lambda$ as geometric objects living on a (geometric) resolution of singularities of $\mathrm{Spec}\, R$.

+ Consider the case X is <u>generic</u>. Then $\mathrm{Spec}\, S = \mathbb{A}^{n^2}$ affine space, thinking of it as $\mathrm{Mat}_{n \times n}(k)$, and $\mathrm{Spec}\, R =$ matrices of rank $\leq m$

Equivalently, $\mathrm{Spec}\, R =$ matrices φ whose image is contained in a subspace of dim m.

Set $\mathbb{G} = \mathrm{Grass}(m,n) = \{ m\text{-planes in } \mathbb{A}^n \}$

Get:

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "q"] \arrow[rr, bend left, "p"] & \mathbb{G} \times \mathrm{Spec}\, S \arrow[r] \arrow[d] & \mathbb{G} \\ \mathrm{Spec}\, R \arrow[r, hook] & \mathrm{Spec}\, S & \end{tikzcd}$$

where $Z = \{ (W, \varphi) : \mathrm{Im}\, \varphi \leq W \}$

<u>Fact</u>: Z is a resolution of singularities of $\mathrm{Spec}\, R$ ("Springer resolution"). $\exists$ similar things in symm./alt. cases.

<u>Theorem</u>: [Kapranov] There is a <u>tilting bundle</u> on $\mathbb{G}$. Let:

$$0 \to K \to \mathcal{O}_{\mathbb{G}}^n \to Q \to 0$$

be a "tautological" exact sequence on $\mathbb{G}$ whose fiber over $W \in \mathbb{G}$ is

$$0 \to W \to k^n \to k^n/W \to 0$$

and set $\mathcal{T} = \bigoplus_{\lambda} \mathbb{S}_\lambda Q^*$

$\lambda \longrightarrow \leq m$ rows and $\leq n-m$ columns

Then: • $\mathcal{T}$ generates $D^b(\mathrm{coh}\, \mathbb{G})$

• $\mathrm{Ext}^i_{\mathbb{G}}(\mathcal{T}, \mathcal{T}) = 0, \ \forall i \neq 0$

$$\Rightarrow \mathrm{RHom}_{\mathbb{G}}(\mathcal{T}, -) : D^b(\mathrm{coh}\, \mathbb{G}) \longrightarrow D^b(\text{mod-}\mathrm{End}(\mathcal{T}))$$

"<u>Proof</u>": Enough cohomology vanishes that $p^*\mathcal{T}$ is still tilting, and that if $M = q_* p^* \mathcal{T}$, then

$\mathrm{End}_R(M) \cong \mathrm{End}_Z(p^*\mathcal{T})$ has finite gldim

(refer to maps p, q in the previous diagram) $\square$