Lecture - 3 : Talk-1 of Minicourse 2
(by Hee Oh).

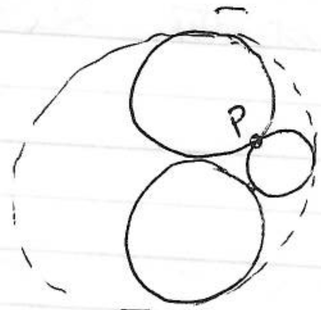
Effective circle count for Apollonian circle packings, via spectral methods :-

Theorem (Apollonians, 200 BC)
Given 3 - mutually tangent circles in the plane, $\exists$ precisely two circles tangent to all three.

Idea (Proof):
through möbius transformation

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$.

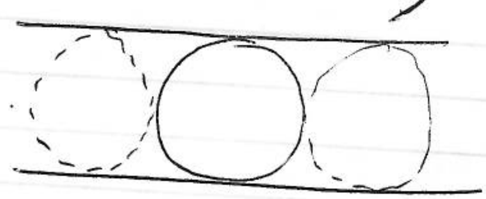
with $ad-bc=1$



· This maps circles(lines) to circles (lines) & preserving angles between them.

$A(p) = \infty$

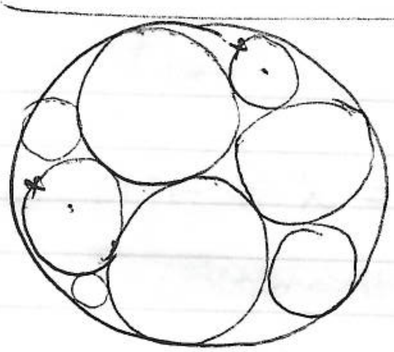
A^{-1}

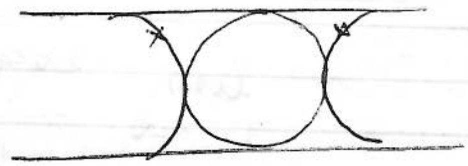


·· Four mutually tangent circle

·Add circles tangent to 3 of the given ones : $\rightarrow$

→ Apollonian Circle packing.

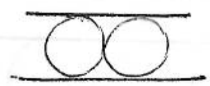




$\mathcal{P}$: ACP.

- If $\mathcal{P}$ is bounded, $N_T(\mathcal{P}) = \#\{C \in \mathcal{P} : \mathrm{curv}(C) < T\} < \infty$.

$$\mathrm{curv}(C) = \frac{1}{\mathrm{radius}(C)}$$

- If $\mathcal{P} =$  , $N_T(\mathcal{P}) = \infty$? in a fixed period.

- Q: $N_T(\mathcal{P}) \sim ?$ as $T \to \infty$.

Definition:- ① $\mathrm{Res}(\mathcal{P}) = \overline{\bigcup_{C \in \mathcal{P}} C}$

$$= \left(\mathbb{C} - \bigcup_{C \in \mathcal{P}} C^{\circ} \right)$$

②. $\alpha :=$ residual dim. of $\mathcal{P}$ = Hausdorff dim. of $\mathrm{Res}(\mathcal{P})$.

α is independent of $\mathcal{P}$. $\alpha = 1{,}30568(8)$, McMullan 98.

Theorem (Boyd 82)

$$\lim_{T\to\infty} \frac{\log N_T(\mathcal{P})}{\log T} = \alpha.$$

$$N_T(\mathcal{P}) \sim c \cdot T^{\alpha} \cdot (\log T)^{\beta} \; ?$$

Theorem: & ① (Kontorovich - Oh) 2009

$$N_T(\mathcal{P}) \sim c \cdot T^{\alpha}.$$

② (Oh - Shah) 2010.

$$N_T(\mathcal{P}) \sim C_A \cdot \mathcal{H}_{\alpha}(\mathrm{Res}(\mathcal{P})) \cdot T^{\alpha}.$$

where $\mathcal{H}_{\alpha}(\mathrm{Res}(\mathcal{P})) = \alpha$ - dim'l H'ff measure of $\mathrm{Res}(\mathcal{P})$.

& C_A = the Apollonian const. is independent of $\mathcal{P}$.

$$\boxed{C_A = \lim_{T\to\infty} \frac{N_T(\mathcal{P})}{\mathcal{H}_{\alpha}(\mathrm{Res}(\mathcal{P})) \cdot T^{\alpha}}}$$

Estimate for C_A ?

③ (Lee - Oh.)

$$N_T(\mathcal{P}) = C_A \cdot \mathcal{H}_\alpha(\mathrm{Res}(\mathcal{P}))\, T^\alpha (1 + O(T^{-\eta}))$$

where $\eta > 0$ is independent of $\mathcal{P}$.

⊛ · $\mathcal{P}$: bounded ~~or~~ unbounded.

$$N_T(\mathcal{P}; E) = \#\{ C \in \mathcal{P} : \mathrm{curv}(C) < T, \; C \cap E \neq \phi \}.$$

· E : bounded.

Theorem (Oh - Shah) : $\partial(E)$: Smooth

$$N_T(\mathcal{P}, E) \sim C_A \cdot \mathcal{H}_\alpha(\mathrm{Res}(\mathcal{P}) \cap E) \cdot T^\alpha.$$

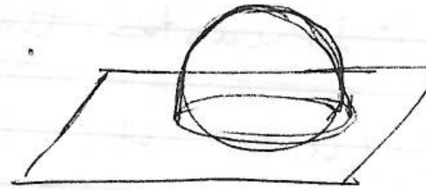
For any E_1 & E_2 bounded with smooth boundary,

$$\frac{N_T(\mathcal{P}, E)}{N_T(\mathcal{P}, E_2)} \sim \frac{\mathcal{H}_\alpha(\mathrm{Res}(\mathcal{P}) \cap E_1)}{\mathcal{H}_\alpha(\mathrm{Res}(\mathcal{P}) \cap E_2)}$$

· $\mathbb{H}^3 = \{ (x_1, x_2, y) \mid y > 0 \}$.

$$ds = \frac{d(\text{Euclidean})}{y}.$$

$\partial(\mathbb{H}^3) = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.



Poincare ext.

- $G = PSL_2(\mathbb{C}) \longleftrightarrow Isom^+(\mathbb{H}^3)$
- $\Gamma < PSL_2(\mathbb{C})$.
- Γ discrete (= Kleinian gps); torsion free.

  not virtually abelian.

① Λ_Γ = the limit set of Γ

= the set of all accumulation point of $\Gamma(z)$, $z \in \hat{\mathbb{C}}$.

② δ_Γ = the critical exp. of Γ

= abs. of conv. for :

$$\sum_{\gamma \in \Gamma} e^{-s d(o, \gamma o)}, \quad o \in \mathbb{H}^3.$$

③ Γ : geom. finite, if $\Gamma \backslash \mathbb{H}^3$ has finite sided ~~finite dimensional~~ fundamental domain.

If Γ is geom. finite,

δ_Γ = H'ff dim. of Λ_Γ.

$\delta_\Gamma = 2 \Leftrightarrow \Gamma < G$ lattice.

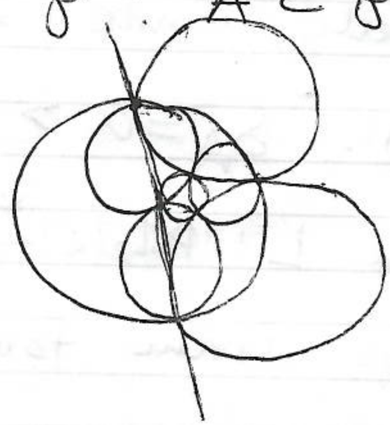
- Descartes' theorem : (1643)

(a, b, c, d) Descartes quadruple

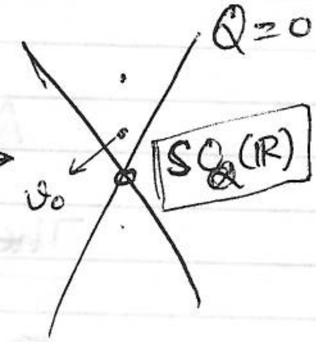
($\Leftrightarrow$ curv. of 4 mutually-tangent circles).

$$\Longleftrightarrow \underset{\substack{\downarrow \\ \text{sign}(3,1)}}{Q(a,b,c,d)} = 2(a^2+b^2+c^2+d^2) - (a+b+c+d)^2 = 0$$

- $\mathcal{P}$: $A \subset \mathcal{P}$.



All Descartes Quadruples associated to $\mathcal{P}$ $\longleftrightarrow$



$A_{\mathcal{P}}$:= the group generated by involutions w.r.t. 4 dual circles ($< PSL_2(\mathbb{C})$)

$$\begin{array}{ccc} PSL_2(\mathbb{C}) & \xrightarrow{\ i\ } & SO_Q(\mathbb{R}) \\ \cup & & \\ A & & \end{array}$$

$A_{\mathcal{P}}(C_1, C_2, C_3, C_4)$.

$v_0 . A$ = All Descartes quadruple associated to $\mathcal{P}$.

- Moreover, $N_T(\mathcal{P}) = \#\text{ of } \{ v \in v_0 . A \mid \|v\|_{max} < T \} \cdot (\#\text{ of } \mathrm{stab}_{\Gamma} v_0) + 3$

$v \cdot A$ is discrete $\Leftrightarrow N_T(\delta) < \infty$

$\Leftrightarrow$ δ bounded on

& vA is geometrically finite &

$\Lambda(A) = \mathrm{Res}(\delta)$. $\delta_A = \alpha > 1$.

<u>Theorem:-</u> Suppose $L : PSL_2(\mathbb{C}) \longrightarrow SO_Q(\mathbb{R})$

Q: any real quadratic form of sign$(3,1)$ with $Q(v_0) = 0$.

· $\Gamma < PSL_2(\mathbb{C})$, geom. finite & $\delta > 1$

· $v_0\Gamma$ is discrete. Then for any norm $\|\cdot\|$ on $\mathbb{R}^4$,

$$\#\{v \in v_0\Gamma : \|v\| < T\} = c \cdot T^{\delta}(1 + O(T^{-\eta})).$$

where η - depends only on the spectral gap for Γ.

<u>Lax-Phillips:-</u> $\delta > 1$, $L^2(\Gamma \backslash \mathbb{H}^3)$.

$\exists$ only finitely many eigenvalues of Δ below $[1, \infty)$:

$$\underbrace{\alpha_0}_{\text{Patterson-Sullivan}} = \underbrace{\delta(2-\delta) < \alpha_1}_{\alpha_1 - \alpha_0 \,=\, \text{spectral gap for } \Gamma} \le \alpha_2 \le \cdots \le \alpha_m < 1.$$

$$N=\left\{\begin{pmatrix}1&z\\0&1\end{pmatrix} : z\in\mathbb{C}\right\}.$$

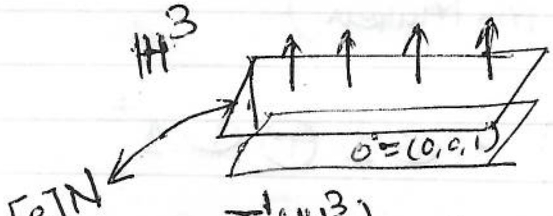
$$M=\left\{\begin{pmatrix}e^{i\theta}&\\&e^{-i\theta}\end{pmatrix}\right\}.$$

$$PSL_2(\mathbb{C}) \curvearrowright \{Q=0\}-\{0\}.$$

$$\|$$

$$NM\backslash PSL_2(\mathbb{C})$$

Understood "$\Gamma\backslash\Gamma N M a_y$" in $\Gamma\backslash G$



$\longleftrightarrow \quad G/K \cong PSU(2)$

$$T^1(\mathbb{H}^3) \longleftrightarrow G/M.$$

$$T^1(\Gamma\backslash\mathbb{H}^3) = \text{"}\Gamma\backslash G/M\text{"}$$

$$a_y=\begin{pmatrix}\sqrt{y}&0\\0&\sqrt{y}^{-1}\end{pmatrix}$$

$$N\cap\Gamma\backslash N = \Gamma\backslash\Gamma N \text{ closed } = \begin{cases}\mathbb{R}^2\\ \mathbb{R}\times\mathbb{R}/\mathbb{Z}\\ \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}.\end{cases}$$

- understand $\Gamma\backslash\Gamma N a_y$ as $y \to 0$

<u>Theorem:-</u> $\delta = 2$, (i.e.) $\Gamma < G$ lattice

$\psi \in C_c^\infty(\Gamma\backslash G/M)$.

$$\int_{N\cap\Gamma\backslash N} \psi(n_z a_y)\,dz = \int_{g\in\Gamma\backslash G/M} \psi\,dg \cdot \frac{\mathrm{vol}(N\cap\Gamma\backslash N)}{\mathrm{vol}(\Gamma\backslash G)} (1+O(y^n)).$$

- Mixing for geod. flow + thickening argument.
  (Eskin – McMullen).

$$\psi \in L^2(\Gamma\backslash G) = \mathbb{C}1 \oplus \mathbb{C}\cdot 1^\perp$$

$$\psi = \langle \psi, 1\rangle 1 + \psi^\perp.$$

$$\int_{N\cap\Gamma\backslash N} \psi(n a_y)\,dn = \int_{\Gamma\backslash G} \psi\,dg \cdot \mathrm{vol}\left(\frac{N}{N\cap\Gamma}\right) + \underbrace{\int \psi^\perp(n a_y)}_{=\,O(y^n)}$$

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$1 < \delta < 2$,

$$L^2(\Gamma\backslash G) = V \oplus V^{\perp}$$

V = unique inf. dim'l irreducible representation of G, where casimir operator C_G acts by scalar $\delta(2-\delta)$.

$$V^M = \bigoplus_{\ell \geq 0} W_\ell^M \quad \leftarrow \text{rep of } K \text{ of dim } 2\ell+1.$$

as $K = PSU(2)$ rep.

$W_\ell^M := \mathbb{C}\,\phi_\ell$, where ϕ_ℓ: real valued function in $C^\infty(\Gamma\backslash G/M)$, with $\|\phi_\ell\|_2 = 1$.

$$\psi = \sum_{\ell \geq 0} \langle \psi, \phi_\ell \rangle \phi_\ell + \psi^{\perp}.$$

$$\int_{N\cap\Gamma\backslash N} \phi_\ell(n_z a_y)\,dz = c_\ell \, y^{2-\delta}, \quad c_\ell \neq 0.$$

<u>Theorem</u>:- $\psi \in C_c^\infty(T^1(\Gamma\backslash \mathbb{H}^3)) = C_c^\infty(\Gamma\backslash G/M)$

$(1 < \delta < 2)$

$$\int_{N\cap\Gamma\backslash N} \psi(n_z a_y)\,dz = \sum_{\ell \geq 0} c_\ell \langle \psi, \phi_\ell \rangle y^{2-\delta} (1 + O(y^{\eta})).$$

Moreover, $\sum_\ell |c_\ell \langle \psi, \phi_\ell \rangle| \leq \underline{S_2(\psi) < \infty}$.