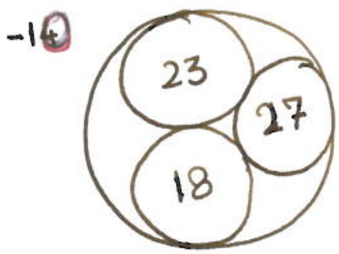
Monday, Feb 6^{th} 2012

Lecture 4: "On the Strong Density Conjecture for Apollonian Circle packings."

By Alex Kontorovich



"bend"

$$b(C) = \frac{1}{r(C)}$$

$r(C)$ = radius of circle C

GLMWY '03

$$v_0 = \begin{pmatrix} -14 \\ 23 \\ 18 \\ 27 \end{pmatrix},$$

Given $\mathcal{P}$, s.t. $\forall\, C \in \mathcal{P}$, $b(C) \in \mathbb{Z}$ define $\mathcal{B} = \{ b(C) : C \in \mathcal{P} \}$.

Theorem (Descartes): If $v = (a, b, c, d)^t$ is the set of bends of a quadruple of mutually tangent circles, then $Q(a, b, c, d) = 2(a^2 + b^2 + c^2 + d^2) - (a+b+c+d)^2 = 0$.

Corollary: Fix $a, b, c \Rightarrow d_\pm$, $d_+ + d_- = 2(a+b+c)$.

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d_+ \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d_- \end{pmatrix},$$

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad S_2, \quad S_3.$$

<u>Definition</u>:- $\Gamma = \langle S_1, \ldots, S_4 \rangle < O_Q(\mathbb{Z})$

<u>Definition</u>:- $\mathcal{O} = \Gamma \cdot v_0 . \Rightarrow \mathcal{B} = \bigcup_{j=1}^{4} \langle e_j, \Gamma \cdot v_0 \rangle .$

<u>Definition</u>:- Let $\mathcal{A} = \{ n \in \mathbb{Z} \mid n \in \mathcal{B} \pmod q, \forall q \geq 1 \}$.

(<u>strong density conjecture (GLMWY)</u>: (Graham Lagarias Mallows Wilks Yan)

$n \in \mathcal{A}$ & $n \gg 1 \Rightarrow n \in \mathcal{B}$.).

<u>Theorem (GLMWY)</u>: $\mathcal{B} \cap [1, N] =: \mathcal{B}(N)$.

$$\# \mathcal{B}(N) \gg N^{1/2}.$$

<u>Proof</u>:- $$M = S_4 \cdot S_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 2 & 2 & -1 & 2 \\ 6 & 6 & -2 & 3 \end{pmatrix}.$$

$$\tilde{M} = J \cdot M \cdot J^{-1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & 2 & 1 & \\ & 4 & 4 & 1 \end{pmatrix}.$$

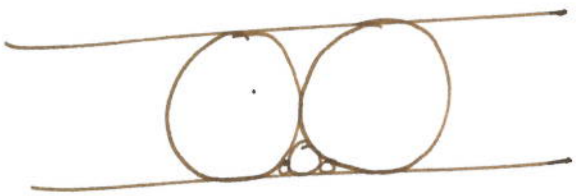
$$\rho : GL_2(\mathbb{R}) \longrightarrow SO(2,1)$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} 1 & & & \\ & \alpha^2 & \cdot & \gamma^2 \\ & \alpha\beta & \cdot & \cdot \\ & \beta^2 & 2\beta\delta & \delta^2 \end{pmatrix}$$

<u>Observation</u>:- $\rho : \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \longmapsto \tilde{M}$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}.$$

$$M^n = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 4n^2+2n & & -2n & 2n+1 \end{pmatrix} \in \Gamma.$$



$$M^n \cdot v_0 = \begin{pmatrix} \vdots \\ \underbrace{4n^2(a+b) + 2n(a+b-c+d)+d}_{An^2+Bn+C} \end{pmatrix} \in \mathcal{O}$$

<u>Theorem (Sarnak 2007)</u>: $\# \mathcal{B}(N) \gg \dfrac{N}{(\log N)^{1/2}}$.

<u>Proof</u>:- $M' = S_3 S_2$.

$$\rho^{-1}(\tilde{M}') = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

<u>Remark</u>:- $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle = \Gamma(2)$

$$\Rightarrow \forall \begin{pmatrix} * & x \\ * & y \end{pmatrix} \in \Gamma(2).$$

$$J^{-1}\rho\begin{pmatrix} * & x \\ * & y \end{pmatrix}\cdot J = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ x^2+y^2+xy-1 & x^2+xy & -xy & xy+y^2 \end{pmatrix}.$$

$$\Rightarrow \quad \mathcal{O} \ni \underbrace{\begin{pmatrix} \\ xy & \cdots \end{pmatrix}}_{\xi_{x,y}} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \vdots \\ Ax^2+Bxy+y^2-a \end{pmatrix} \quad (\in \Gamma).$$

$$\Rightarrow \quad \mathcal{B} \supset \{Ax^2+Bxy+Cy^2-a\}.$$

(Definition :- $J^{-1}\rho(\Gamma(2))\cdot J = \overset{= SO(2,1)}{\Gamma_1} < \Gamma$).

Theorem: (Fuchs '2010) : $\# \mathcal{B}(N) \gg \dfrac{N}{(\log N)^{0.15\ldots}}$

Theorem (Bourgain - Fuchs, 2011) :

$$\# \mathcal{B}(N) \gg N.$$

Conjecture: $\# \mathcal{B}(N) = \# \mathcal{A}(N) + O_\rho(1)$.

Fuchs : $\# \mathcal{A}(N) = \dfrac{\# \mathcal{A}(24)}{24} \cdot N + O(1)$.

Theorem (Bourgain - K '2012)

$$\# B(N) = \# \mathcal{A}(N) + O_\rho(N^{1-\varepsilon}).$$

- In dim n, Soddy - Gossett.
- $n \geq 4$, no integrality, no packing.
- $n = 3$.

<u>Theorem (K '11)</u>: For Soddy sphere packings,

$$\# \mathcal{B}(N) = \# \mathcal{A}(N) + O_{\mathcal{P}}(1).$$

<u>Proof</u>: Same procedure gives values of quadratic forms in 4-variables.

$(\Gamma > \Gamma_1 = \Gamma_0(\mathcal{O}_3, \sqrt{3}))$.

<u>Theorem (K - Oh '09)</u>:

$$\# \{ C \in \mathcal{P} \mid b(C) < T \} \sim c \cdot T^{\delta}.$$

<u>Observation</u>: For Zaremba, $G_A = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} \middle| a \leq A \right\rangle$.

(semigroup)

$\langle e_2, G_A e_2 \rangle \in$ full?

- <u>Ingredients</u>: ① Major arcs, need work of I. Vinogradov on bisector counts in ∞-volume 3-manifolds.

(extending Bourgain - K - Sarnak '10).

<u>Theorem</u>: (<u>Vinogradov</u>): $G = SL_2(\mathbb{C}) = K A^{t} K$, $A = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix}$

$K = SU(2) = MHM$, $H = SO(2)$, $M = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$.

$K/M \cong S^2 = \partial \mathbb{H}^3$. and $G = MH \cdot \underset{a}{A}^t \underset{m}{M} \underbrace{HM}_{K/M \ni s_2^{-1}}$

Take $\Phi, \Psi \subset K/M$, $I \subset [0,1]$, as $T \to \infty$,

$$\sum_{\gamma \in \Gamma} \mathbb{1} - \left\{ \begin{array}{l} s_1 \in \Phi \\ |a| \leq \cancel{X} T \\ m \in I \\ s_2^{-1} \in \Psi \end{array} \right\} = C \cdot \mu(\Phi) \cdot \mu(\Psi) \cdot \ell(I) T^{2\delta} \cdot \left(1 + O(T^{2\delta - \varepsilon})\right)$$

- Minor arcs: Use recent work of Bourgain '11. where he proves, $\# \mathcal{B} \cap [1, N] \cap \text{Prime} \gg \frac{N}{(\log N)}$.

- Definition:- $$R_N(n) = \sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| \approx T}} \sum_{x, y \in X} \mathbb{1}\{ n = \langle e_{\cancel{n}\,4}, \xi_{x,y} \cdot \gamma \cdot v_0 \rangle \}$$

$$N = X^2 \cdot T, \quad T = N^{\cancel{1/100}\ 1/100},$$

Observation:- $R_N(n) > 0 \Rightarrow n \in \mathcal{B}$.

$$\hat{R}_N(\theta) = \sum_{\cancel{\delta}\,\gamma} \sum_{x,y} e(\theta \langle e_n, \xi_{x,y} \cdot \gamma \cdot v_0 \rangle).$$

$$R_N(n) = \int_0^1 \hat{R}_N(\theta)\, e(-n\theta)\, d\theta.$$

$$= \int_{\mathcal{M}} + \int_{m} = \mathcal{M}_N(n) + \mathcal{E}_N(n).$$

If $\theta = \frac{r}{q}$, $\quad \hat{R}_N\left(\frac{r}{q}\right) = \sum_{x,y} \sum_{\gamma \in \Gamma} e_q\left(r \langle e_n, \xi_{x,y}\gamma \cdot v_0 \rangle\right)$

Theorem: $\quad \mu_N(n) \gg \sigma(n) \cdot \dfrac{X^2 T^{\delta}}{X^2 T}, \quad \Gamma = \Gamma(q) \cdot \Gamma(q) \backslash \Gamma.$

$\left(\hat{R}_N(0) \simeq X^2 T^{\delta} \right).$

$\Big($ If $\quad |\mathcal{E}_N(n)| = o\left(T^{\delta-1}\right)$

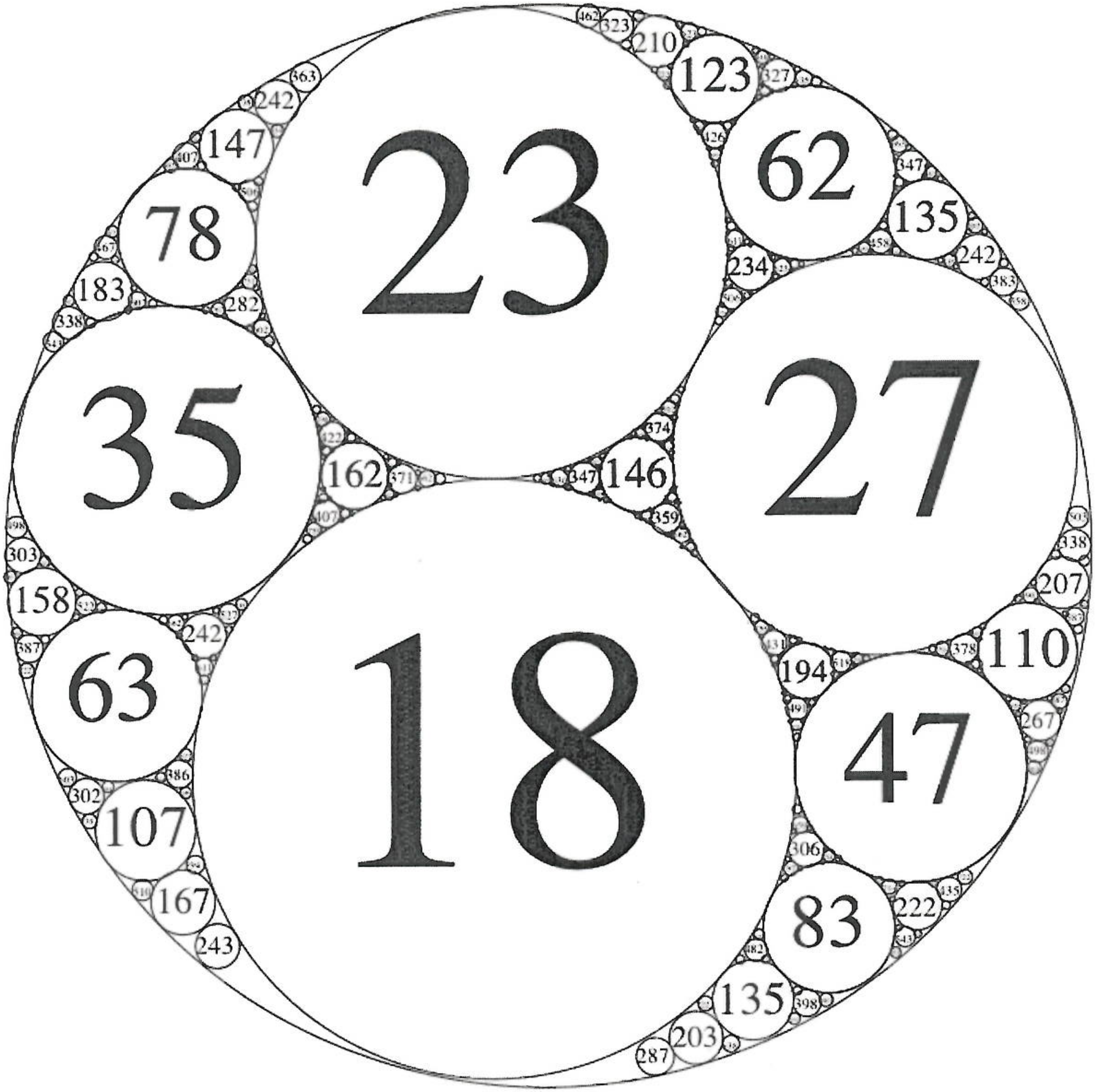
$\Rightarrow$ strong density conjecture $\Big)$.

Theorem: $\quad \sum_n |\mathcal{E}_N(n)|^2 = \int_{\mathfrak{m}} |\hat{R}_N(\theta)|^2 \, d\theta$

$$\ll X^4 T^{2\delta} \big/ N^{1+\varepsilon}.$$

- Look at: $\quad \displaystyle\sum_{\substack{n<N \\ n \in A}} 1 \ll \sum_{n<N} \frac{|\mathcal{E}_N(n)|^2}{T^{2\delta-2}} \ll \frac{X^4 T^{2\delta}}{N^{1+\varepsilon} T^{2\delta-2}} = N^{1-\varepsilon}.$

$\left(|\mathcal{E}_N(n)| \gg T^{\delta-1} \right)$



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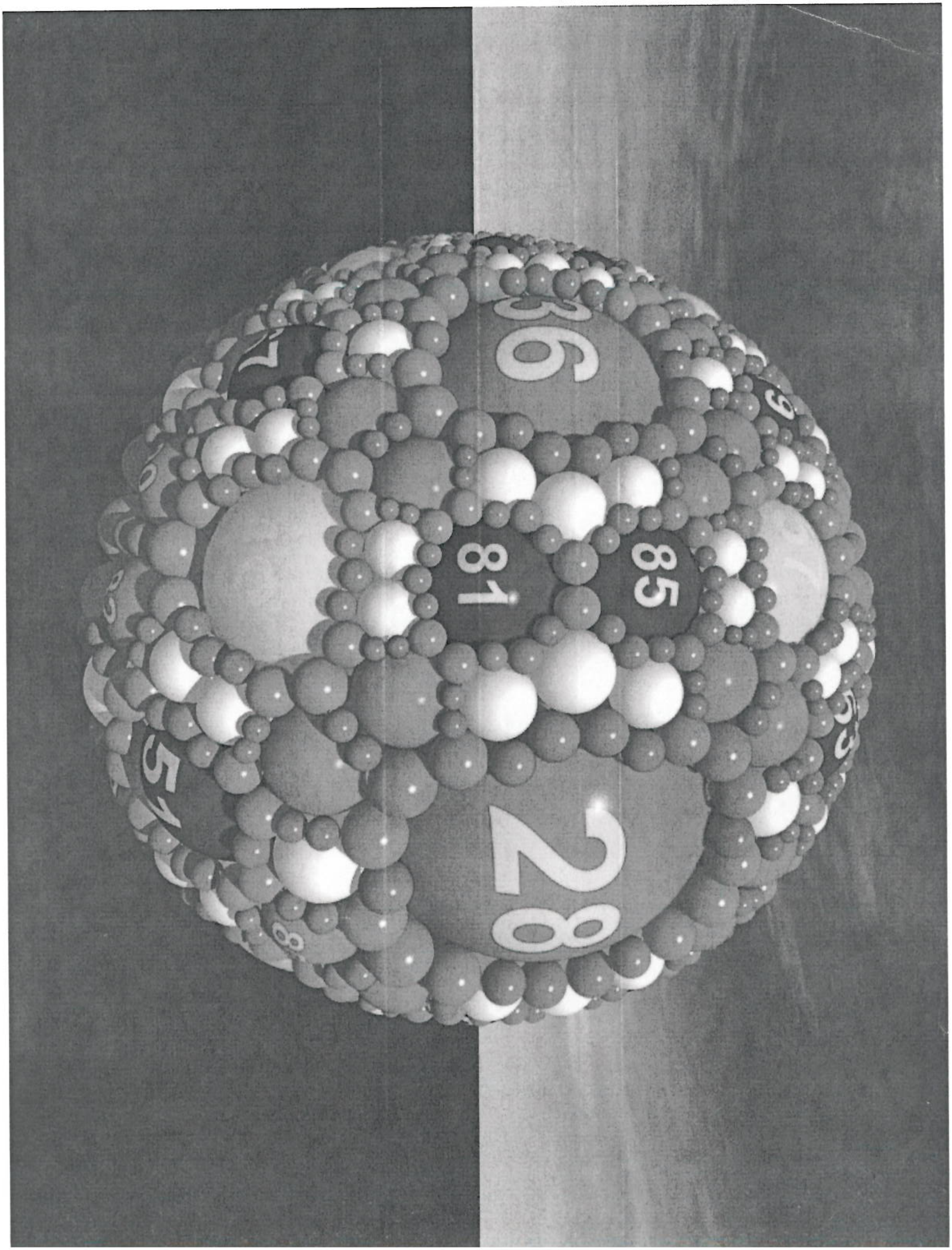
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