Lecture: 8 Tuesday, Feb 7^{th}, 2012

Talk 2 of Mini Course ~~1~~ by Hee Oh

" ~~Effective circle count for Apollonian circle-packings, via spectral methods~~ "

(Joint with N. Shah)

- $G = SO(n,1)^\circ \curvearrowright V \ni v_0$
  $\hookrightarrow$ irreducible finite dimensional-representation.

- Assume $v_0 G$ is a symmetric variety.
  $$v_0G = H\backslash G, \quad H = \text{stab}_G v_0 \left(= SO(k,1) \times SO(n-k) \quad 0 \le k \le n-1 \right).$$

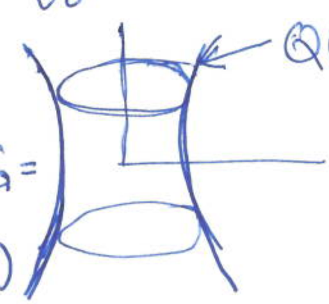
Example: Q: real quadratic form of sign $(n,1)$.

$SO(Q) \curvearrowright \mathbb{R}^{n+1} \ni v_0$ standard representation.

" $Q(v_0) > 0$

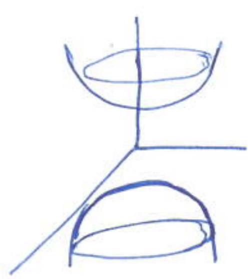
$Q(v_0) = m$ " $m>0$

$v_0 G =$



$H = SO(n-1,1)$

$Q(v_0) < 0$



$H = SO(n)$.

$\Gamma < G$, $v_0\Gamma$ discrete
discrete gp.

<u>Theorem</u> (Duke-Rudnick-Sarnak, Eskin-McMullen '93)

$\|\cdot\|$ norm on V. If $\mathrm{vol}(\Gamma\backslash G) < \infty$ &

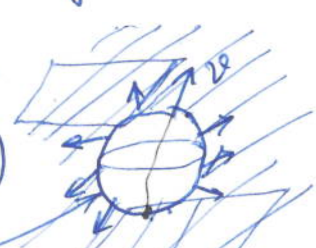
$\mathrm{vol.}(H\cap\Gamma\backslash H) < \infty$; $\#(v_0\Gamma \cap B_T) \sim \dfrac{\mathrm{Vol}(H\cap\Gamma\backslash H)}{\mathrm{vol}(\Gamma\backslash G)} \cdot \mathrm{Vol}(B_T)$

$$B_T = \{ v \in v_0 G \mid \|v\| < T \}.$$

- $G = HAK$, $A = \{a_t\}$ one parameter gp consisting of diagonalizable elements
  K: maximal compact subgp.

- $G = SO(n,1)^\circ = \mathrm{Isom}^+(\mathbb{H}^n)$.
  $M = C_K(A)$, $N = \{ g \in G \mid a_t g a_{-t} \to e \text{ as } t \to \infty \}$
  = expanding horospherical subgp.

$G/K \longleftrightarrow \mathbb{H}^n$, $G/M \longleftrightarrow T^1(\mathbb{H}^n)$



(~~Joint work with N. Shah~~).

N-orbits in $T^1(\mathbb{H}^n)$.

The main ingredient of the proof of the above theorem is (3).

Theorem: $\psi \in C_c(T^1(\Gamma\backslash \mathbb{H}^n)) = C_c(\Gamma\backslash G/M)$ $= C_c(\Gamma\backslash G)^M$. the following equi-distribution on translates of an H-orbit.

~~Suppose~~

Suppose $Vol(\Gamma\backslash G)<\infty$, $Vol(H\cap\Gamma\backslash H)<\infty$.

$$\lim_{t\to\infty} \int_{H\cap\Gamma\backslash H} \psi(ha_t)\,dh = \frac{Vol(H\cap\Gamma\backslash H)}{Vol.(\Gamma\backslash G)} \int_{\Gamma\backslash G} \psi\, dg.$$

~~Meaning of~~

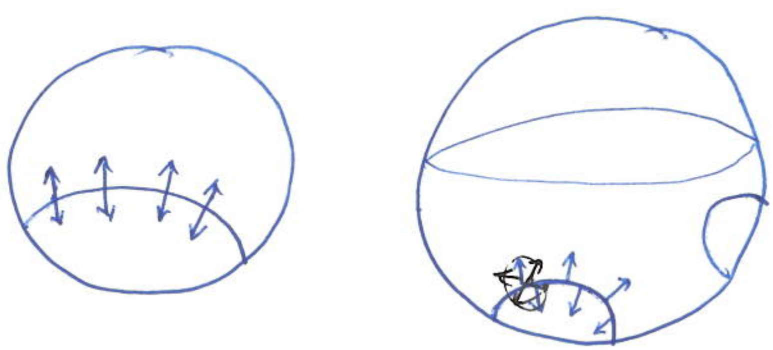
$1 \in L^2(\Gamma\backslash G)$
mixing of good flow for Haar measure

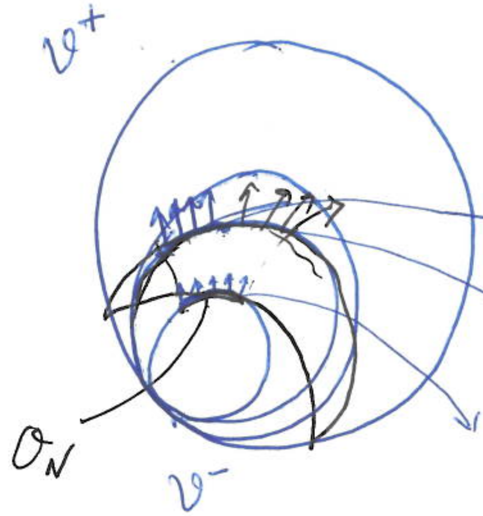
uniq N-invariant ergodic measure on $\Gamma\backslash G/M$, which is not ~~closed~~ supported on a closed N-orbits.

$H = SO(K,1)\times SO(n-K)$ in $G/M = T^1(\mathbb{H}^n)$.

$\Downarrow$ unit normal bundle of $\mathbb{H}^K$ in $T^1(\mathbb{H}^n)$

Example. $n=2$, $\mathbb{H}^2$





$\mathcal{O}_H a_t$

$$\mathcal{O}_N a_t = a_t \underbrace{(a_{-t}\, \mathcal{O}_N a_t)}_{\text{"}N\text{" as } t\to\infty}$$

- If Γ is not a lattice,
  - * m^{Haar} is not mixing for geodesic flow
  - * m^{Haar} is <u>not</u> N-ergodic.

- $\Gamma < G$, discrete gp, not virtually abelian.

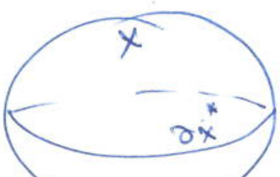
$\{\nu_x \mid x \in \mathbb{H}^n\}$: Γ-invariant conformal density of dim "δ".

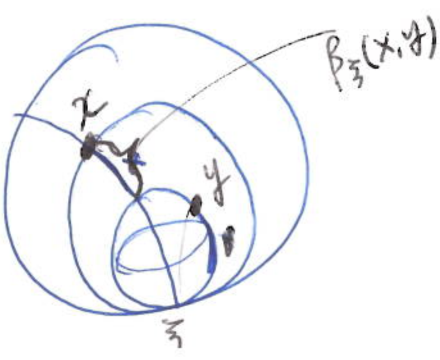
(finite - positive measure on $\partial_\infty(\mathbb{H}^n)$). supported on $\Lambda(\Gamma)$

① $\gamma_* \nu_x = \nu_{\gamma(x)}$ $\forall \gamma \in \Gamma$

② $\dfrac{d\nu_x}{d\nu_y}(\xi) = e^{\delta \beta_\xi(y,x)}$

signed ~~density~~ ~~between~~ distance between horospheres based at ξ passing through x & y





- If Γ is a lattice, $\{m_x : x \in \mathbb{H}^n\}$ (5)

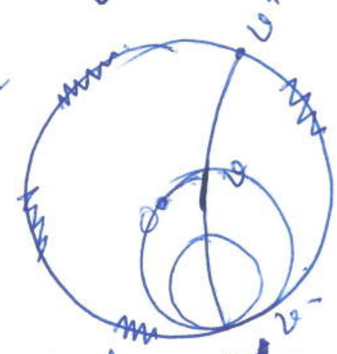
(m_x: probability measure on $\partial(\mathbb{H}^n)$ which is invariant $K_x = \mathrm{stab}_G x$.) is a $\underline{\Gamma\text{- inv.}}$ conformal density of $\dim = (n-1)$.

- Patterson - Sullivan $\Rightarrow$ gave a construction of Γ- inv. conformal - density of dim δ_Γ = critical exp. of Γ.

- Fix $o \in \mathbb{H}^n$. $T^1(\mathbb{H}^n) \longleftrightarrow (\partial\mathbb{H}^n \times \partial\mathbb{H}^n - \mathrm{Diag}) \times \mathbb{R}$

$$v \longleftrightarrow (v^+, v^-, \beta_{v^-}(o, v))$$

(1) m^{BMS} = Bowen - Margulis - Sullivan measure on $T^1(\Gamma\backslash\mathbb{H}^n)$



$$d\tilde{m}^{BMS}(v) = e^{\delta\beta_{v^+}(o,v)} e^{\delta\beta_{v^-}(o,v)} \cdot d\nu_o(v^+) \cdot d\nu_o(v^-) \cdot ds$$

* definition is independent of $o \in \mathbb{H}^n$.

* Γ- invariant

* $\underbrace{\{g^t\}}_{\text{geodesic flow}}$ - invariant.

$\Rightarrow m^{BMS}$ on $T^1(\Gamma\backslash\mathbb{H}^n)$.

- $\text{supp}\, m^{BMS} = \Gamma \backslash \{ v \in T^1(\mathbb{H}^n) : v^{\pm} \in \Lambda_\Gamma \}$

  $\subset$ convex core of Γ

- If Γ is convex cocompact ($\Leftrightarrow$ convex core of Γ is compact) then $|m^{BMS}| < \infty$.

- If Γ is geometrically finite ($\Leftarrow$ unit nbd of convex core has finite vol.), $|m^{BMS}| < \infty$. (Sullivan)

Theorem:- (Rudolph '83, Babillot 02) If $|m^{BMS}| < \infty$, then m^{BMS} is mixing for the geodesic flow.

$\psi_1, \psi_2 \in C_c(T^1(\Gamma \backslash \mathbb{H}^n))$,

$$\int \psi_1(g a_t)\, \psi_2(g)\, dm^{BMS}(g) \xrightarrow[t \to \infty]{} m^{BMS}(\psi_1) \cdot m^{BMS}(\psi_2).$$

(2) m^{BR} = Burger- Roblin measure.

$$d\tilde{m}^{BR}(v) = e^{(n-1)\beta_{v^+}(0,v)}\, e^{\delta \beta_{v^-}(0,v)}\, dm_0(v^+)\, d\nu_0(v^-)\, ds$$

$\hookrightarrow$ Γ- invariant & N- invariant _ m^{BR} on $T^1(\Gamma \backslash \mathbb{H}^n)$.

* m^{BR} is the unique "N"-invariant & ergodic on $T^1(\Gamma \backslash \mathbb{H}^n)$ which is

not supported on closed N-orbits. (7)

$\underline{\Gamma : \text{discrete}}$ $\Rightarrow |m^{BR}| = \infty \Leftrightarrow \Gamma$ is not a lattice.

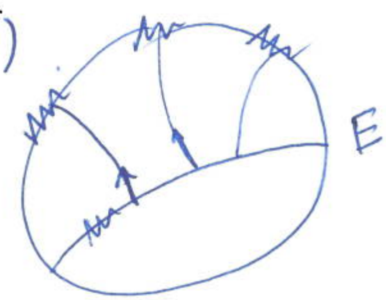
(Oh-Shah).

(3). μ_E^{PS} ; $\tilde{E}$ = the unit normal bundle to $\mathbb{H}^k$.

$(H = SO(k,1) \times SO(n-k)$.

$E = P(\tilde{E})$ under $T^1(\mathbb{H}^n) \xrightarrow{P} T^1(\Gamma \backslash \mathbb{H}^n)$.

& $\mu_{\tilde{E}}^{PS}(v) = e^{\delta \beta_{v^+}(0,v)} d\nu_0(v^+)$
$(v \in \tilde{E})$
$\hookrightarrow H \cap \Gamma$ - invariant.



$\Rightarrow$ get a measure on E : μ_E^{PS}.

<u>Theorem</u> (Oh-Shah) $|m_\Gamma^{BMS}| < \infty$, $|\mu_E^{PS}| < \infty$.

$$\#(v_0 \Gamma \cap B_T) \sim \frac{|\mu_E^{PS}|}{|m^{BMS}|} \cdot \frac{T^{\delta/\lambda}}{\delta \cdot \|\hat{v}_0\|^{\delta/\lambda}}$$

λ : highest wt of L

$\hat{v}_0$ = proj. of v_0 to the highest wt sp.

(<u>Thm</u> (Oh-Shah)
$\delta >$ parabolic co-rank of Γ_H
$\Updownarrow$
$|\mu_E^{PS}| < \infty$.)

$\max (\text{rank } \Gamma_\xi - \text{rank}(\Gamma_{\dots}))$

$$\left(\mathrm{Vol}(B_T) \sim \frac{T^{\frac{(n-1)}{\lambda}}}{(n-1)\,\|\hat{v}_0\|^{(n-1)/\lambda}} \right.$$

<u>Theorem</u> (O.-Shah): $\psi \in C_c(T^1(\Gamma\backslash\mathbb{H}^n))$ $\quad \Gamma\backslash\Gamma H \overset{\text{closed}}{\subset} \Gamma\backslash G$

$$e^{(n-1-\delta)t} \int_{H\cap\Gamma\backslash H} \psi(ha_t)\,dh \longrightarrow \frac{|\mu_E^{PS}|}{|m^{BMS}|} \cdot m^{BR}(\psi).$$

<u>Remark</u> $|\mu_E^{PS}| < \infty$ if

1) Γ is convex cocompact; or

2) $\delta > n-k$; or

3) $\mathrm{vol}(H\cap\Gamma\backslash H) < \infty$.