Lecture: Thursday - Feb 9th, 2012

Title: 3-manifold Groups, Surface groups + thin-subgps of $SL_2(\mathbb{Z})$ by

(Based on joint work with D.D. Long &

D.D. Long + M. Thistlethwaite).

3-manifolds
↓
π_1 (compact or. 3-manifolds).

surface gps := π_1 (closed orientable surface)

Two main themes:-

1. Subgp ~~of~~ structure of $SL_3(\mathbb{Z})$.
   - Both finite index.
   - infinite index.

(Remark:- $SL_3(\mathbb{Z})$ has C.S.P.)

2. Representations of 3-manifold groups and surface groups.

- $$\begin{array}{ccc} SL_3(\mathbb{Z}) & \xrightarrow{\varphi_n} & SL_3(\mathbb{Z}/n\mathbb{Z}) \\ {\scriptstyle f.i.}\downarrow & & \\ \Gamma & & \\ \downarrow & & \\ \Gamma(n) & & \end{array}$$

<u>Question:</u> (Lubotzky)

Does $SL_3(\mathbb{Z})$ contain "arbitrarily" small 2-generator subgp of finite-index.

i.e. if $\Gamma < SL_3(\mathbb{Z})$ has finite index (f.i),

then $\Gamma \underset{f.i}{\supset}$ 2-gen-gp.

(Its theorem of Sharma + Venkataramana that the above holds with 2 replaced by 3).

· what do infinite index finitely generated Zariski-dense subgroup look like ?

- there are Zariski-dense free subgps.
- there are surface group $<$ Zariski-dense and not.

<u>Questions</u>: ① (Serre) Is $SL_3(\mathbb{Z})$ coherent ?

(i.e. Is every finitely generated subgp finitely presented?) $(n=3)$.

② Does $SL_3(\mathbb{Z})$ have finitely generated intersection property?

(If $H, K < SL_3(\mathbb{Z})$ are finitely generated, Is $H \cap K$ finitely generated.)

- Representations of 3-manifold groups and surface groups has been a powerful tool in low dimensional topology.

<u>classical</u>. Reps $\pi_1(\Sigma_g) \longrightarrow PSL_2(\mathbb{R})$
(Teich. theory)
$\longrightarrow PSL_2(\mathbb{C})$

<u>Thurston</u>: Reps $\pi_1(M^3) \longrightarrow PSL_2(\mathbb{C})$.

<u>outgrowth</u>: Culler - Shalen $\pi_1(M^3) \longrightarrow PSL_2(\mathbb{C})$.

Casson: $\pi_1(\Sigma_g)$, $\pi_1(M^3) \longrightarrow SU(2)$, $SO(3)$.

Kronhe~~ssier~~imer - Mrowka: $\pi_1(M^3) \longrightarrow SU(2)$, $SO(3)$.

<u>More recently</u>:

$\pi_1(\Sigma_g) \longrightarrow SL_n(\mathbb{R})$

$\pi_1(M^3) \longrightarrow SL_n(\mathbb{R})$ $(n=4)$

$SU(2,1)$, $SU(3,1)$.

<u>Today</u>: Reps of $\pi_1(M^3)$, $\pi_1(\Sigma_g) \longrightarrow SL_3(\mathbb{R})$.

- Approach to establishing $SL_3(\mathbb{Z})$ does not have finitely generated intersection property. (f.g.i.p).

- Hyperbolic 3-manifold $= \mathbb{H}^3/\Gamma$,

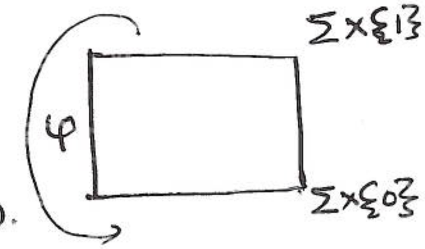
  $\Gamma < PSL_2(\mathbb{C})$: discrete torsion free.

Suppose that $\exists M$ a finite volume hyperbolic 3-manifold that fibres over S^1 and

$\pi_1(M) \xrightarrow{1-1} SL_3(\mathbb{Z})$.

M fibres means $M = \Sigma \times I / \sim$;

where $\varphi : \Sigma \longrightarrow \Sigma$ self homeomorphism.



Here,

$$1 \longrightarrow \pi_1(\Sigma) \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z} \longrightarrow 1$$

$$\pi_1(\Sigma) = F.$$

Such $\pi_1(M)$ do not have finitely generated intersection property.

<u>why</u> $\pi_1(M) = \langle F, t \rangle$

$F \triangleleft \pi_1(M)$, $H = \langle g^N, t^N \rangle$, take $g \in F$

$H \cap F \neq 1$, $H \cap F \triangleleft H$ ⟵ infinitely generated.

Results:-

Theorem 1 (LR) $\exists$ 2-gen, finite index - subgps $\{N_j\} \subset SL_3(\mathbb{Z})$ s.t. $\bigcap N_j = 1$ (uses Representation theory of a 3-manifold gp).

Theorem 2: $\exists$ an infinite family of non-conjugate, Zariski-dense, infinite index, freely indecomposable, purely semisimple, subgps of $SL_3(\mathbb{Z})$ that are all isomorphic to a fixed group G that is word-hyperbolic and has property FA.

(* Rest of the talk is on slides)

$$\Gamma = \pi_1(S^3 \setminus \infty) = \left\langle x, y, z \;\middle|\; \begin{array}{l} zxz^{-1} = xy \\ zyz^{-1} = yxy \end{array} \right\rangle.$$

(figure eight)

Discussion of the proof of Theorem 1

SL(3, R)-representations of the figure-eight knot group

Let K = the figure-eight knot and $\Gamma = \pi_1(S^3 \setminus K)$.

Γ admits a presentation coming from the fact that $S^3 \setminus K$ is a once-punctured torus bundle over S^1.

Let generators for the fibre group F be x and y and let z be the stable letter.

Then Γ is presented as:

$$< x, y, z | z.x.z^{-1} = x.y, \quad z.y.z^{-1} = y.x.y >.$$

Remark Γ is 2-generator generated by z, x.

Γ has $Sl_3(\mathbb{R})$-reps.

1. Γ has $Sl_2(\mathbb{R})$ and $SU(2)$ reps (irreducible) coming from analyzing the real points of its $Sl_2(\mathbb{C})$-char variety

$$Sl_2(\mathbb{R}),\, SU(2) \hookrightarrow Sl_3(\mathbb{R})$$

1. One particular $SO(3)$ rep arises from $\Gamma \twoheadrightarrow A_4$

$$x \longmapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$y \longmapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\Big\}$ generate $\mathbb{Z}/_2 \times \mathbb{Z}/_2$

$$z \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(has order 3).

$$\leq SL_3(\mathbb{Z})$$

Given a rep^{n} $\rho : \Gamma \longrightarrow G$
(Lie gp)
can try and deform it.

Start looking for infinitesimal obstructions to small defs.

Computational tools developed by Cooper-Long-Thistlethwaite in

① "Flexing closed hyperbolic manifolds"
② "Computing varieties of repn's of hyperbolic 3-manifolds into $SL_4(\mathbb{R})$"

Can be used to experiment

ie try to deform

$$\Gamma \twoheadrightarrow A_4 \leq SL_3(\mathbb{R})$$

Define a map $\beta_T : \Gamma \longrightarrow \mathrm{SL}(3, \mathbf{Z}[\mathrm{T}])$ by

$$\beta_T(x) = \begin{pmatrix} -1+T^3 & -T & T^2 \\ 0 & -1 & 2T \\ -T & 0 & 1 \end{pmatrix}$$

$$\beta_T(y) = \begin{pmatrix} 1 & 0 & 0 \\ -T^2 & 1 & -T \\ T & 0 & -1 \end{pmatrix} \qquad \beta_T(z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & T^2 \\ 0 & 1 & 0 \end{pmatrix}$$

Can check by matrix muliplication that β_T is a homomorphism.

$$\beta_T(zyz^{-1}) = \begin{pmatrix} -1+T^3 & T & 0 \\ -T^5 & -1+T^3 & 0 \\ -T+T^4 & -T^3 & 1 \end{pmatrix} = \beta_T(yxy)$$

$$\beta_T(zxz^{-1}) = \begin{pmatrix} 1+T^3 & -T & 0 \\ 3T^2 & -1 & -T \\ 2T & 0 & -1 \end{pmatrix} = \beta_T(xy)$$

<u>Note</u>. When $T=0$, get the rep to A_4 defined above. Can think of β_T as a "<u>deformation of this rep to 1st order</u>."

Some comments about the reps β_T

1. The representations β_T are never faithful.

Amazingly the following relation holds in $\beta_T(\Gamma)$ (indeed $\beta_T(F)$) for every value of the parameter T ($X = \beta_T(x)$, $Y = \beta_T(y)$):

$$\boxed{\begin{gathered} X^{-1}YX^{-1}YX^{-1}X^{-1}YYYXYYXY^{-1}X = \\ XY^{-1}XYYXYYYX^{-1}X^{-1}YX^{-1}YX^{-1} \end{gathered}}$$

2. $\beta_T(\Gamma)$ contains unipotent elements for all $T \neq 0$.

EG. $\beta_T(y^2) = \begin{pmatrix} 1 & 0 & 0 \\ -T^2 & 1 & 0 \\ -2T & 0 & 1 \end{pmatrix}$

Other $\beta_T((x^{-1}y)^2)$ (has char poly $(-1+a)^3$)
$(\neq 1)$

Now focus on the case when $\boxed{T > 0 \textbf{ is an integer.}}$

By Margulis's normal subgroup theorem, $\beta_T(\Gamma)$ has finite index if and only if $\beta_T(F)$ has finite index, and henceforth we deal with $\beta_T(F)$.

Theorem 1 follows from:

Claim: *With $T > 0$ as above, $\beta_T(F)$ has finite index in* $\mathrm{SL}(3, \mathbf{Z})$ *and* $\bigcap_{T>0} \beta_T(F) = 1$.

The proof of the claim will follow from showing for those T as above, $\beta_T(F)$ is Zariski dense in $\mathrm{SL}(3, \mathbf{R})$ and the following result of Venkataramana

Theorem:(Venkataramana) *Suppose that $n \geq 3$ and $x \in SL(n, \mathbf{Z})$ is a unipotent matrix such that $x - 1$ has matrix rank 1.*

Suppose that $y \in \mathrm{SL}(\mathrm{n}, \mathbf{Z})$ is another unipotent such that x and y generate a free abelian group N of rank 2.

Then any Zariski dense subgroup of $\mathrm{SL}(\mathrm{n}, \mathbf{Z})$ *containing N virtually, is of finite index in* $\mathrm{SL}(\mathrm{n}, \mathbf{Z})$.

We have already seen that $\beta_T(F)$ contains unipotents.

Need to ensure they have unipotents as in Venkataramana's Thm.

Let $X_T = \beta_T(x)$ and $Y_T = \beta_T(y)$.

We have the following unipotent elements in $\beta_T(F)$.
(by calculation!)

$$b_1 = X_T^{-1}.Y_T.Y_T.Y_T.X_T.Y_T.Y_T.X_T.Y_T^{-1}.X_T$$

$$b_2 = X_T.Y_T^{-1}.X_T.Y_T.Y_T.X_T.Y_T.Y_T.Y_T.X_T^{-1}$$

We can make things more visible.

Let $P = \begin{pmatrix} 0 & 1 & 1 \\ 2T & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$$c_1 = P^{-1} b_1 P = \begin{pmatrix} 1 & 0 & -T^2(-1+2T)(-5+3T^3) \\ 0 & 1 & -T(-1+2T)(-2+3T^3) \\ 0 & 0 & 1 \end{pmatrix}$$

$$c_2 = P^{-1} b_2 P = \begin{pmatrix} 1 & 0 & -3T^2(-1+2T) \\ 0 & 1 & -T(-1+2T)(-2+3T^3) \\ 0 & 0 & 1 \end{pmatrix}$$

<u>Note</u> - $[c_1, c_2] = 1$

<u>Check</u>: $c_1 - 1$, $c_2 - 1$, have rank $\stackrel{?}{=} 1$, $\langle c_1, c_2 \rangle \stackrel{?}{\cong} \mathbb{Z} \oplus \mathbb{Z}$

Showing Zariski density

A key ingredient is the following result of Lubotzky (special case where with $n = 3$).

Proposition:(Lubotzky) *Let* $G < \mathrm{SL}(3, \mathbf{Z})$ *be generated by* $A \subset \mathrm{SL}(3, \mathbf{Z})$. *Assume that for some odd prime* $p \geq 3$, *the image of* A *in* $\mathrm{SL}(3, \mathrm{p})$ *under the reduction modulo* p *surjects* G *onto* $\mathrm{SL}(3, \mathrm{p})$. *Then* G *is a Zariski dense subgroup of* $\mathrm{SL}(3, \mathbf{R})$.

We apply this in the following way with our next result.

Theorem 3:(Long-R) *Let G be a finitely generated non-solvable subgroup of* $\mathrm{SL}(3,\mathbf{Z})$. *Suppose that there is an element $g \in G$ whose characteristic polynomial is $\mathbf{Z}$-irreducible and non-cyclotomic.*
Then for infinitely many primes p, reduction modulo p surjects G onto $\mathrm{SL}(3,\mathrm{p})$.

Zariski density is then proved by exhibiting explicit elements.

EG. The char poly. of $\beta_T([x,y])$ is

$$Q^3 - (3-2T^3+T^6)Q^2 - (-3+10T^3-4T^6)Q - 1$$

Can check for $T \neq 0$, this is non-cyclotomic, irreducible.

Can also check that $\beta_T(F)$ contains a free non-abelian gp. (eg using unipotent elts)

Another family of representations

Define a map $\rho_k : \Gamma \longrightarrow \mathrm{SL}(3, \mathbf{Z}[\mathrm{k}])$ by

$$\rho_k(x) = \begin{pmatrix} 1 & -2 & 3 \\ 0 & k & -1-2k \\ 0 & 1 & -2 \end{pmatrix},$$

$$\rho_k(y) = \begin{pmatrix} -2-k & -1 & 1 \\ -2-k & -2 & 3 \\ -1 & -1 & 2 \end{pmatrix}, \quad \rho_k(z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -k \\ 0 & 1 & -1-k \end{pmatrix}$$

Then ρ_k is a homomorphism. These are also Zariski dense, when $k \in \mathbb{Z}$.

- When $k = 0, 2, 3, 4, 5$ $\rho_k(\Gamma)$ has finite index.
- $\rho_1(\Gamma)$ is likely virtually free

<u>Qn</u> For other values of k is $\rho_k(\Gamma)$ thin?
Are there unipotents?

Coherence

Suppose that for some $k \in \mathbf{Z}$ we can arrange that:

- $\rho_k(F)$ *is of infinite index in* $\rho_k(\Gamma)$
- $\rho_k(F)$ *is not free*
- *the virtual cohomological dimension of* $\rho_k(\Gamma)$ *is* 2.

Then $\mathrm{SL}(3, \mathbf{Z})$ *is not coherent.*

Proof: It is a theorem of Bieri that in a group of cohomological dimension 2, any finitely presented normal subgroup is free or it is of finite index.

Apply this to $\rho_k(\Gamma)$.

$\rho_k(F)$ is not free and infinite index in $\rho_k(\Gamma)$; i.e. we have an infinite index normal subgroup of $\rho_k(\Gamma)$ that is finitely generated but not free.

By passing to a torsion-free subgroup of finite index Δ_k in $\rho_k(\Gamma)$ it follows from standard properties of cohomological dimension that Δ_k has cohomological dimension 2. The only possibility from Bieri's result is that $F_k \cap \Delta_k$ is not finitely presented. This completes the proof.

Remarks on surface groups

Similar methods can be used to show:

[ie deform a Fuchsian rep^n in the Hitchin cpt]

Family of reps of (3,3,4) triangle gp

$$a \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 2-t+t^2 & 3+t^2 \\ 0 & -2+2t-t^2 & -1+t-t^2 \\ 0 & 3-3t+t^2 & (-1+t)^2 \end{pmatrix}$$

are discrete faithful $\forall t \in \mathbb{R}$.

Theorem 2: (Long-R-Thistlethwaite) *There exists an infinite family of non-conjugate Zariski dense, purely semisimple subgroups of* SL(3, Z) *isomorphic to the* (3, 3, 4) *triangle group.*

One corollary of this is:

Corollary: *There exists an infinite family of non-conjugate Zariski dense, infinite index, freely indecomposable, purely semisimple subgroups of* $\mathrm{SL}(3, \mathbf{Z})$ *isomorphic to a fixed group G which is word hyperbolic and has Property FA.*

This is in contrast to word hyperbolic groups, Mapping Class groups.

"Purely semisimple" = diagonalizable over $\mathbb{C}$.

Final Comments

1. *Does there exist an orientable finite volume hyperbolic 3-manifold M for which $\pi_1(M)$ admits a faithful representation into* $\mathrm{SL}(3, \mathbf{Z})$*?*

Indeed, one might ask:

1'. *Does there exist a compact orientable hyperbolizable 3-manifold M which is not an I-bundle over a surface and for which $\pi_1(M)$ admits a faithful representation into* $\mathrm{SL}(3, \mathbf{Z})$*?*

Or replace $\mathbf{Z}$ by $\mathbf{R}$ and require discreteness.

2. *Does there exist a compact orientable hyperbolizable 3-manifold M which is not an I-bundle over a surface and for which $\pi_1(M)$ admits a faithful discrete representation into* $\mathrm{SL}(3, \mathbf{R})$*?*