LINEAR GROUPS

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JOINT WORK WITH
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BUDAPEST

BREUILLARD - GREEN-TAO PROVED INDEPENDENTLY
PY - SZABÓ (EFFECTIVE) IN 2010

$G = SL(n, q)$ $\quad \langle A \rangle = G \Rightarrow$

EITHER $\quad A^3 = G$

OR $\quad |A^3| \gg |A|^{1+\varepsilon}$

FOR SOME $\quad \varepsilon = \varepsilon(n)$

IN FACT

BOTH PROOFS EXTEND TO SIMPLE GROUPS OF LIE TYPE OF BOUNDED RANK

FOR $SL(2,p)$, $SL(3,p)$ THIS IS A FAMOUS RESULT OF HELFGOTT

FOR $SL(2,q)$, DINAI, VARJÚ

$q = p^\alpha$, p prime

BUILDING BLOCK FOR DIAMETER AND EXPANSION ESTIMATES

GROW IN LINEAR GROUPS

?

HE + B-G-T PROBLEM:

TEST CASE : $S \subseteq SL(n, p)$

POLYNOMIAL INVERSE THEOREM (P_y-S_z) (2010)

$S \subseteq SL(n, p)$, SYMMETRIC SUBSET

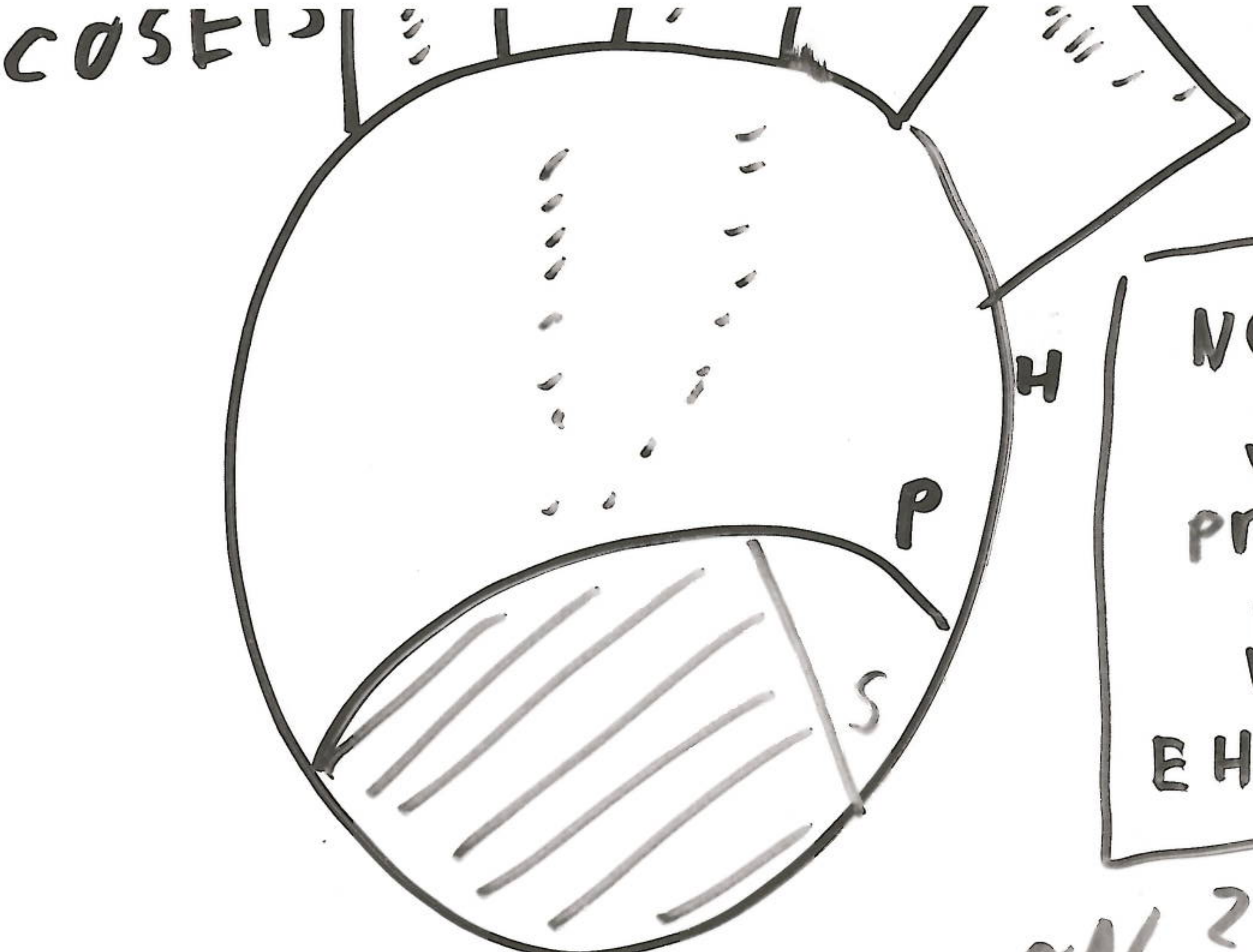
WITH $|S^3| \leq K|S|$ $K \geq 1$

$\Rightarrow$ THERE ARE SUBGROUPS

$H \geq P$ NORMALISED BY S

SUCH THAT

- P IS PERFECT (i.e $P' = P$)
- H/P IS $\boxed{\text{SOLUBLE}}$
- $P \subseteq S^6$ (S ESSENTIALLY CONTAINS P)
- S IS COVERED BY $K^{c(n)}$ (POLYNOMIALLY MANY) COSETS OF H (H ESSENTIALLY CONTAINS S)



NORI

$\Downarrow$

proof

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EHK

HOW TO GO ON?

MORE PRECISE RESULT:

GILL-HELFGOTT: H/P NILPOTENT

MORE GENERAL RESULT

<u>PCL INVERSE THEOREM (Py-Sz, 2011)</u>

THE SAME HOLDS

FOR $S \leq SL(n,q)$, $q = p^{\alpha}$

BREUILLARD - GREEN - TAO THEOREM IN CHAR = 0

$S \subseteq GL(n, \mathbb{C})$ S SYMMETRIC

$|S^3| \leq K|S| \Rightarrow$ S CAN BE COVERED BY $K^{C(n)}$ COSETS OF A SOLUBLE SUBGROUP H.

BY EARLIER WORK OF BREUILLARD - GREEN IN FACT H CAN BE TAKEN TO BE NILPOTENT

THE PROOF USES THE FACT THAT IN CHAR = 0 A VIRTUALLY SOLUBLE GROUP Γ HAS A SOLUBLE SUBGROUP H WITH $|\Gamma / H| \leq f(n)$ - WHICH IS NOT TRUE IN CHAR = $p > 0$

$S \subseteq GL(n, \mathbb{F})$, $\mathbb{F}$ AN ARBITRARY FIELD, S SYMMETRIC

$|S^3| \leq k|S| \Rightarrow S$ CAN BE COVERED BY $k^{c(n)}$ COSETS OF A SUBGROUP Γ WITH A SOLUBLE NORMAL SUBGROUP N SUCH THAT

$\Gamma / N \cong L_1 \times \ldots \times L_m$ WHERE L_i ARE FINITE SIMPLE GROUPS OF LIE TYPE IN CHARACTERISTIC $= \mathrm{char}(\mathbb{F})$ + EXTRA INFO

EARLIER HRUSHOVSKI PROVED SUCH A RESULT WHERE THE NUMBER OF COSETS IS $\leq$ SOME $f(k, n)$

THE POLYNOMIAL INVERSE THEOREM HOLDS FOR $S \subseteq SL(n, \mathbb{F})$, $\mathbb{F}$ AN ARBITRARY FIELD

THE PROOF USES THE PREVIOUS SPECIAL CASES INCLUDING THE PRODUCT THEOREM

BREUILLARD-GREEN-TAO

INVERSE THEOREM (2011)

1) IN ARBITRARY GROUPS !!!
2) MUCH BETTER DESCRIPTION
3) BUT ONLY BOUNDS OF THE FORM $f(K)$

TASK: PROVE A FINITE BY NILPOTENT VERSION OF THE POL INVERSE THEOREM (SEE TAO'S BLOG)

$S \subseteq GL(n, \mathbb{F})$, $\mathbb{F}$ ARBITRARY

$|S^3| \leq k|S| \implies S$ IS

COVERED BY $k^{O(n)}$

COSETS OF A

"NICE" = VIRTUALLY SOLVABLE

GROUP Γ

- SIMILAR TO THE POLYNOMIAL FREIMAN-RUZSA CONJECTURE
- BUT SWEEPS ALL FINITE OR ABELIAN EXAMPLES UNDER THE RUG

PRODUCT THEOREM

$G = SL(n, q) \qquad |G| \approx q^{n^2-1}$

T MAXIMAL TORUS e.g.
A FULL DIAGONAL SGROUP

$|T| \approx q^{n-1} = |G|^{\frac{1}{n+1}}$

<u>HELFGOTT</u> IF $\langle A \rangle = SL(n, q)$

$|A^3| \approx |A| \Rightarrow$

$|A \cap T| \lessapprox |A|^{\frac{1}{n+1}}$

+ SIMILAR RESULTS FOR SOME OTHER SUBGROUPS

QUESTION: WHAT IS THE RIGHT GENERALISATION?

MAIN TOOL IN THE PROOF: <u>LARSEN-PINK TYPE INEQUALITY</u>

$SL(n,q) \subset SL(n,\overline{\mathbb{F}}_p) \supset V$ VARIETY

$$|A \cap V| < |A^k|^{\dim(V)/\dim(G)}$$

CHEATING!! THERE ARE SOME CONSTANTS WHICH DEPEND ON $\dim(V)$ AND $\deg(V)$ k DEPENDS ON $\dim(V)$

THE Py-Szabó VERSION IS EVEN WEAKER (BUT EQUALLY USEFUL)

<u>QUESTION</u> WHAT ARE THE „INTERESTING" CHOICES FOR V?

PAIRS $C_G(x)$ AND $cl(x)$

HERE THE COSETS OF $C_G(g)$ ARE THE FIBERS OF THE MAP $g \to g^{-1} x g$

$$\dim(C_G(x)) + \dim(\overline{cl(x)}) = \dim(G)$$

$\Rightarrow$ THE UPPER BOUND FOR $|A \cap \overline{cl(x)}|$ IMPLIES A SHARP LOWER BOUND FOR $|A \cap C_G(x)|$

(IF A DOES NOT GROW)

e. g. T MAX TORUS

T_r = REGULAR SEMISIMPLE ELEMENTS IN T (ALL EIGEN VALUES DISTINCT)

CONJUGATE AND WATCH THE TORI

LEMMA a'la Helfgott

$$\frac{|A^k \cap H|}{|A^2 \cap H|} \leq \frac{|A^{k+1}|}{|A|}$$

i.e. GROWTH IN A SUBGROUP IMPLIES GROWTH IN G.

HELFGOTT: THERE IS A RICH T

IF ALL CONJUGATES OF T ARE RICH $\Rightarrow$ A IS „VERY LARGE"

OTHERWISE WE HAVE A RICH T AND $a \in A$ SUCH THAT T^a IS POOR

FOR A^3 T^a IS RICH

$\Rightarrow$ WE FOUND GROWTH IN T^a.

DICHOTOMY IF A DOES NOT GROW (i.e. $|A^3| \sim |A|$)

$\Rightarrow$ a) IF $A \cap T_r = \emptyset \Rightarrow$

$|A \cap T| \ll |A|^{n-2/n^2-1}$

b) IF $A \cap T_r \neq \emptyset \Rightarrow$

$|A \cap T| \gg |A|^{n-1/n^2-1}$

i.e. $A \cap T$ IS EITHER "SMALL" OR "LARGE"

"POOR T" "RICH T"

QUESTION: HOW DO WE BREAK THE BALANCED DISTRIBUTION OF A IN G?

ARBITRARY

$|A^3| \sim |A|$ $\Rightarrow$ FIRST (BIG) STEP

Th A IS COVERED BY

$K^{C(n)}$ COSETS OF A

VIRTUALLY SOLUBLE GROUP

PROOF GENERALISATION

- OF THE LARSEN-PINK STUFF
- OF THE CONJUGATING TRICK

THIS IS DIFFICULT - WE USE "GENERALISED TORII" IN "NOT NECESSARILY SIMPLE LINEAR ALGEBRAIC GROUPS

WE OBTAIN GROWTH IN SOME SUBGROUP AND USE THIS INDUCTIVELY

$\Rightarrow$ ---------- .

THE PROOF USES
THE PRODUCT THEOREM
+ MANY TRICKS e.g.
GOWERS (SEE N_1-P_4)

G FINITE GROUP,
$\deg_{\mathbb{C}}(G) \geq k$, α, β, γ
SUBSETS OF G WITH

$$|\alpha||\beta||\gamma| > \frac{|G|^3}{k} \Rightarrow$$

$$\Rightarrow \alpha\beta\gamma = G$$

($\deg_{\mathbb{C}}(G)$ = MIN DEG OF
A COMPLEX
REPRESENTATION

LEMMA

$1 \in A = A^{-1}$, $|A^3| < K|A|$,

$N \lhd G$, $N \leq A^k$

AND $\deg_{\mathbb{C}}(N) \geq K^{3k}$ (N IS SUFFICIENTLY QUASI RANDOM)

$\Rightarrow A^6 \geq N$

HENCE A WEAKER VERSION OF THE POL INVERSE THEOREM IMPLIES THE STRONGER VERSION

$A \subseteq SL(n, \mathbb{F})$, $|A^3| < K|A|$

WE ALREADY KNOW THAT A IS IN POLYNOMIALLY MANY COSETS OF A VIRTUALLY SOLUBLE G

<u>MALCEV</u> Γ finitely gen

$\Gamma \subseteq GL(n, \mathbb{F})$ char $\mathbb{F} = p$

FOR EVERY $g_1, \ldots g_t \in \Gamma$ $\exists$ FINITE FIELD $k = p^\alpha$ AND A HOMOMORPHISM $\phi : \Gamma \to GL(n, k)$ SUCH THAT $\phi(g_1), \ldots, \phi(g_t)$ ARE DISTINCT

$\Rightarrow$ WE CAN USE THE POL INVERSE THEOREM FOR FINITE FIELDS FOR QUOTIENTS OF $\langle A \rangle$ TO COMPLETE THE PROOF