Lecture : ~~Thursday~~ Friday , Feb 10^{th}, 2012 ①.

How generic are thin Groups ?

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Expander Graphs $\longrightarrow$ Number Theory
- Thin groups ← Harder
- not thin gps → classical

Definition: $\{X_i\}$ → infinite family of k-regular connected finite-graphs.

adj. Matrix $M_i \xrightarrow[\text{eigenvalues}]{} k = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_s \geq -k$.

$\{X_i\}$ is an expander family if $\lim_{n\to\infty} \lambda_1(M_n) < k$.

Theorem (Salehi - Golsefidy / Varjú 2011)

$\Gamma \subseteq GL_n(\mathbb{Z})$, $\Gamma = \langle S \rangle$ ← finite.

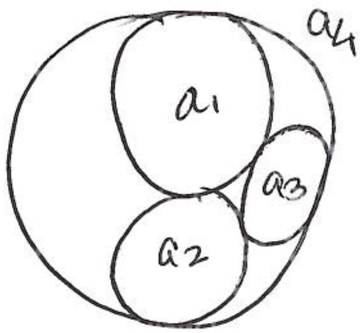
necessary + sufficient condition for
$\{ Cay(\Gamma/d, S) \mid d > 1 \text{ square free} \}$ to be
an expander family is that the connected

component of $Zcl(\Gamma)$ is perfect.

Definition:- $\Gamma \subset GL_n(\mathbb{Z})$. $G = Zcl(\Gamma)$

Γ is thin if $[G(\mathbb{Z}) : \Gamma] = \infty$.

App 1: Integer Apollonian Packings.



→ curvatures $\frac{1}{r}$, a_1, a_2, a_3, a_4.

→ If $a_1, a_2, a_3, a_4 \in \mathbb{Z}$ then all circles in packing have curvatures in $\mathbb{Z}$.

→ There are infinitely many primitive integer ACP's.

- $ACP\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}\right) \xrightarrow{\text{correspond}}$ orbits of $\Gamma \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$.

(quadruples of mutually tangential circles)

$\Gamma \subset O_Q(\mathbb{Z}) \subseteq GL_n(\mathbb{Z})$.

Q — Descartes form.

- Γ is thin

Theorem:- (Fuchs '2010) For a primitive integer ACP P, let $\mathcal{O}(P)$ be corresponding

Γ orbit. Then e —

$$Zcl\left(\left\{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathcal{O}(P) \;\middle|\; abcd \text{ is } \overset{\text{expect } G.}{\overline{28}}\text{-almost prime}\right\}\right) = Zcl(\mathcal{O}).$$

- App 2 :- (Ellenberg - Hall - Kowalski - 2010)

Theorem (corollary 4 in E-H-K)

K : number field, $f \in K[x]$ is a square-free polynomial degree $2g$.

$\mathcal{U}$ - complement of 0's of f in $\mathbb{A}^1$

$\mathcal{C}/\mathcal{U}$: $y^2 = f(x)(x-t)$ ← not 0 of f.

$$J_t = Jac(\mathcal{C}_t)$$

$$\forall d \geq 1, \bigcup_{[K_1, K]=d} \{ t \in \mathcal{U}(K_1) \mid End_{\mathbb{C}}(J_t) \neq \mathbb{Z} \} \text{ is finite.}$$

expanders ⟶ consider image ——

$\Gamma \leq Sp_{2g}(\mathbb{Z})$ of monodromy rep.

associated to $\mathcal{C}$.

- here Γ is finite index $Sp_{2g}(\mathbb{Z})$. (J-K, YU '1995) so not thin.

If $g > 1 \to Sp_{2g}(\mathbb{Z})$ has Kazhdan property(T)

Γ has T $\longrightarrow$ Γ has expander property

(has finite index in $Sp_{2g}(\mathbb{Z})$) (Margulis '73)

- App 1 $\to$ thin ← how generic?

  App 2 $\to$ not thin.

<u>Remark</u>:- $\Gamma = \langle S \rangle$ ← finite $\to$ deciding if Γ thin, (<u>HARD</u>)

but $Zcl(\Gamma) \longrightarrow$ doable.

- $G = SL_n(\mathbb{Z})$: $g = (g_1, \dots, g_k) \in G^k$
  $(n \geq 2)$. ($k \geq 2$ <u>fixed</u>)

$\Gamma(g) = \langle g_1, \dots, g_k, g_1^{-1}, \dots, g_k^{-1} \rangle$.

<u>Question</u>:- relative to same sequence of probability measures μ_y, how probable is $[G : \Gamma(g)] = \infty$?

μ_Y

(1) Choose (finite set of) generators of G.

$\mu_Y \rightarrow$ counting measures for words of length Y in generators of G.

(Question answered by R. Aoun 2010
I. Rivin 2010).

(2) μ_Y is the counting measures on G in a "ball" of radius Y,
(+ inverses).

$$\|\gamma\| = \lambda_{max}(\gamma^t \gamma) \;;\; (\gamma \in G).$$

Theorem:- (Fuchs - Rivin, 2012)

Let $G = SL_n(\mathbb{Z})$. Let $\Gamma(g)$ as before, μ_Y as in (2), —

$$\lim_{Y \to \infty} \mu_Y\left(\{g = (g_1, g_2) \in G^2 \mid [G : \Gamma(g)] = \infty\}\right) = 1.$$

"How $\underset{??}{\underline{\text{generic}}}$ are thin monodromy gps? (6)

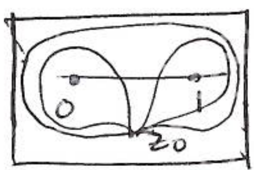
Examples of Monodromy groups (Beukers-Heckman'89)

Consider $D(\alpha,\beta) = D(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$.

in $\mathbb{Q}$.

$$\left(\text{for } \theta = z\frac{d}{dz} \right) \quad = \prod_{i=1}^{n} (\theta + \beta_i - 1) - z \prod_{i=1}^{n} (\theta + \alpha_i).$$

$H(\alpha,\beta)$ – monodromy group associated to $D(\alpha,\beta)$.



$\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.

Theorem (Levelt)

$$P_1(x) = \prod_{j=1}^{n} (x - e^{2\pi i \alpha_j})$$

$$= x^n + A_1 x^{n-1} + \cdots + A_n.$$

$$P_2(x) = \prod_{j=1}^{n} (x - e^{2\pi i \beta_j}) = x^n + B_1 x^{n-1} + \cdots + B_n.$$

$$H(\alpha,\beta) = \left\langle \begin{bmatrix} 0 & 0 & \cdots & -A_n \\ & & & -A_{n-1} \\ & I & & \vdots \\ & & & -A_1 \end{bmatrix}, \begin{bmatrix} 0 & \cdots & 0 & -B_n \\ & & & -B_{n-1} \\ & I & & \vdots \\ & & & -B_1 \end{bmatrix} \right\rangle$$

restriction : $P_1 + P_2$ are products of cyclotomic polynomials.

Beuckers- Heckman $\rightarrow$ $H(\alpha, \beta)$ is either

- finite
- if n is odd & H is infinite
  $Zcl(H) = O_n(\mathbb{C})$.
- If n is even and H infinite
  - $\hookrightarrow$ $A_n/B_n = 1 \Rightarrow Zcl(H) = Sp_n(\mathbb{C})$
  - $\hookrightarrow$ $A_n/B_n = -1 \Rightarrow Zcl(H) = O_n(\mathbb{C})$.

I) H is symplectic (C - GR).

example from Chen - Yang - Yui, 2008.

e.g. $$H = \left\langle \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 9 & 9 & 1 & 0 \\ 0 & -6 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle \subseteq Sp_4(\mathbb{Z}).$$

So, wat we can do?

find candidate : congruence subgp $H_c \subseteq Sp_4(\mathbb{Z})$

s.t. either $H = H_c$ or H thin.

II) orthogonal groups

II a) Signature $(n-1, 1)$.

- these $H(\alpha, \beta)$'s fall into 9 explicit families.

- A :- finite order.

- $S = A^{-1}B$ is a reflection.

- $\langle A^{-k} S A^{k} \rangle$ finite ~~order~~ index in $\langle A, B \rangle$.

* $f \xrightarrow{\text{integer form of}} \text{sign}(n-1, 1)$

$O_f(\mathbb{Z}) \longrightarrow R_f(\mathbb{Z})$ ← generated by all hyperbolic reflections in $O_f(\mathbb{Z})$.

Nikulin '1981 : All but finitely many $R_f(\mathbb{Z})$'s are thin.

<u>Theorem</u> (<u>Vinberg -1984</u>)

$\nexists$ discontinuous groups of motions of $\mathbb{H}^n$ generated by reflections with compact finite volume fundamental domain if $n \geq 30$.

<u>Theorem</u> (Prokhorov '86):

noncompact case.

none $n \geq 995$.

Mikulin '86 $n \geq 30$.