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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Shuangjian Zhang **Email/Phone:** shuangjian.zhang@mail.utoronto.ca

Speaker's Name: Sun-Yung Alice Chang

Talk Title: Higher order isoperimetric inequalities - an approach via the method of optimal transport

Date: 08 / 22 / 2013 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: isoperimetric inequality, optimal transport, higher order, k-convex, Monge-Ampere Equation

Please summarize the lecture in 5 or fewer sentances: This lecture provides a proof of higher order isoperimetric inequalities, by using method of optimal transport.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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① Isoperimetric inequality:

$L^2 \geq 4\pi A$, for $\Omega \subseteq \mathbb{R}^2$, bounded.

(here $L = |\partial\Omega|$, $A = |\Omega|$).

$|\Omega|^{\frac{1}{n}} \leq \bar{C}_n |\partial\Omega|^{\frac{1}{n-1}}$, for $\Omega \subseteq \mathbb{R}^n$, bounded

$\bar{C}$ means sharp, i.e. only attained when Ω = ball.

②. Higher order isoperimetric inequality:

Minkoviski mixed volume:

$$\mathrm{Vol}(\Omega + tB) = \sum_{k=0}^{n} W_k(\Omega) t^k \qquad \text{(here } B \text{ is unit ball)}$$

$$W_0(\Omega) = |\Omega|$$

$$W_1(\Omega) = C_{n,1} |\partial\Omega|$$

$$\vdots$$

$$W_k(\Omega) = C_{n,k} \int_{\partial\Omega} \sigma_{k-1}(L_{\partial\Omega})\, d\sigma$$

(here $L = L_{\partial\Omega}$ = second ~~fund~~ fundamental form of $\partial\Omega$)

for A $n \times n$ matrix $A \sim \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

Then $\sigma_\ell(A) = \ell$-th elementary symmetric function of (λ_i)

e.g. $\sigma_1(A) = \sum \lambda_i = \mathrm{Tr}(A)$

$$\sigma_2(A) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left[(\lambda_1 + \cdots + \lambda_n)^2 - \sum_{i=1}^{n} \lambda_i^2\right] = \frac{1}{2}\left(|\mathrm{Tr}A|^2 - |A|^2\right)$$

$$\vdots$$

If $L_{\partial\Omega} = A$, then $\sigma_1 =: H$, $2\sigma_2 = H^2 - |L|^2$

③. Alexsandrov – Fenchel Inequality

$(**)_\ell$ $\left(\oint_{\partial\Omega} \sigma_{\ell-1}(L_{\partial\Omega})\, d\sigma \right)^{\frac{1}{n-\ell}} \leq \bar{C}_{n,\ell} \left(\int_{\partial\Omega} \sigma_\ell(L_{\partial\Omega})\, d\sigma \right)^{\frac{1}{n-\ell-1}}$

for $0 \leq \ell \leq n$.

e.g. for $\ell = 0$. $\oint_{\partial\Omega} \sigma_{-1}$ is $|\Omega|$

e.g. for $\ell = 1$, the inequality is $|\partial\Omega|^{\frac{1}{n-1}} \leq \bar{C}(\oint_{\partial\Omega} H)^{\frac{1}{n-2}}$.

④. • Huisken –

$(**)_1$ holds when Ω is "outwards minimizing", i.e.

$\Omega' \supseteq \Omega \Rightarrow |\partial\Omega'| \geq |\partial\Omega|$

• P.F. Guan – Li

Def: Ω is k-convex, if $L_{\partial\Omega} \in \Gamma_k^+$, i.e.

$\sigma_\ell(L_{\partial\Omega}) > 0$ for all $\ell \leq k$.

e.g. n-convex is convex

Thm. If Ω is ~~k convex~~ star-shaped domain in $\mathbb{R}^n$. with k-convex boundary. then $(**)_\ell$ holds for all $1 \leq \ell < k$. With the equality holds if and only if Ω is a ball.

⑤ $(*)_l \quad |\Omega|^{\frac{1}{n}} \leq \bar{C}_{n,k} \left(\oint_{\partial\Omega} \sigma_k (L_{\partial\Omega}) \right)^{\frac{1}{n-k-1}}$

Thm 1 (Yi Wang, Alice Chang)

$(*)_k$ holds for $k=1,2$, when assumption Ω is $(k+1)$-convex.

Note: "Qian": $(*)_k$ holds for all $1 \leq k \leq n$.

Thm 2 (Yi Wang, Alice Chang)

If Ω is $(k+1)$-convex then $(**)_l$ holds for all $l \leq k$.

Note: here is not sharp constant.

⑥. Optimal Transport: $D, D' \subseteq \mathbb{R}^n$, $T: D \to D'$,

$\mu = f\,dx$, $\nu = g\,dy$ are the density of D, D', respectively.

Then the problem is to find:

$$\inf_{T_{\#}\mu = \nu} \int_D |x - T(x)|^2 \, dx$$

(here, cost function $c(x,y) = |x-y|^2$)

"Brenier": T_{min} is achieved when $T(x) = \bar{\nabla}\phi$, where ϕ is convex and $\bar{\nabla}$ means Euclidean dir.

Monge-Ampère equation: $\det(\bar{\nabla}^2 \phi) = \dfrac{f(x)}{g(\bar{\nabla}\phi(x))}$

⑦ Application: $|\Omega|^{\frac{1}{n}} \le C_n \, |\partial\Omega|^{\frac{1}{n-1}}$ (*)

Let $T: \Omega \to B$ be the optimal transport.

$T = \nabla\phi$, where ϕ is convex.

Let $f = \frac{\chi_\Omega}{|\Omega|}$, $g = \frac{\chi_B}{|B|}$.

Then
$$\left(\frac{|B|}{|\Omega|}\right)^{\frac{1}{n}} |\Omega| = \int_\Omega \left(\frac{|B|}{|\Omega|}\cdot \chi_\Omega\right)^{\frac{1}{n}} = \int_\Omega \left(\det(\nabla^2\phi)\right)^{\frac{1}{n}}$$

$$\le \int_\Omega \frac{\Delta\phi}{n} = \frac{1}{n}\oint_{\partial\Omega} \frac{\partial\phi}{\partial n} \le \frac{1}{n}|\partial\Omega|$$

$\Rightarrow$ (*). (Notice that $\left(\det(\nabla^2\phi)\right)^{\frac{1}{n}} \le \frac{\Delta\phi}{n}$.

⑧. Pf of Thm1.

$$\left(\det(\nabla^2\phi)\right)^{\frac{2}{n}} \le \frac{\sigma_2(\nabla^2\phi)}{C_n^2}$$

$$\Rightarrow \quad 2\int_\Omega \sigma_2(\nabla^2\phi)\,dx \le \oint_{\partial\Omega} H$$

$\Rightarrow$ (*) $_{|}$

9. Background

$$2\sigma_2(\bar\nabla^2\phi) = (\bar\Delta\phi)^2 - (\bar\nabla^2\phi)^2 = ((\bar\Delta\phi)\delta_{ij} - \bar\nabla^2\phi_{ij})\bar\nabla^2_{ij}\phi$$

for A $n\times n$ matrix,

$$2\sigma_2(A) = T_1(A)_{ij}A_{ij}.$$

$$T_1(A) = -A_{ij} + \sigma_1(A)\delta_{ij}$$

$$\sum_i \partial_i(T_1(A_{ij})) = 0 \quad \forall j.$$

So, $2\sigma_2(\bar\nabla^2\phi) = \partial_i((T_1)_{ij}(\bar\nabla^2\phi)\phi_j)$.

$$A\in\Gamma_2^+ \Rightarrow T_1(A)\geq 0$$

Hence $2\oint_\Omega \sigma_2(\bar\nabla^2\phi)\,dx = \oint_{\partial\Omega}(T_1)_{ij}(\bar\nabla^2\phi)\bar\nabla_i\phi\,\nu_j\,d\mu$

$\overset{?}{\leq} \oint_{\partial\Omega} H\,d\mu.$ (here $\bar\nabla\phi$: transport maps)

Gauss equation:

$1\leq i,j\leq n$, $1\leq\alpha,\beta,\gamma$ tangential on $\partial\Omega \leq n-1$.

$\bar\nabla^2_{\alpha,\beta}\phi = \nabla^2_{\alpha,\beta}\phi + L_{\alpha\beta}\phi_\nu$ (here ϕ_ν: outward normal)

$\bar\nabla^2_{\alpha\nu}\phi = \nabla_\alpha\phi_\nu - L_{\alpha\gamma}\phi_\gamma$

$$\nabla^2\phi = \left(\begin{array}{c|c} \bar\nabla^2_{\alpha\beta}\phi & \bar\nabla^2_{\alpha\nu}\phi \\ \hline \bar\nabla^2_{\alpha\nu}\phi & \phi_{\nu\nu}\end{array}\right) = A+B, \text{ where } A = \left(\begin{array}{c|c} \nabla_{\alpha\beta}\phi & \nabla_\alpha\phi_\nu \\ \hline \nabla_\alpha\phi_\nu & \phi_{\nu\nu}\end{array}\right)$$

$$LHS = \oint_{\partial\Omega}(T_1)_{ij}(\bar\nabla^2\phi)\bar\nabla_i\phi\,\nu_j = \underbrace{2\oint_{\partial\Omega}\Delta\phi\,\phi_\nu}_{I_1} + \underbrace{\oint H\phi_\nu^2 + \oint L_{\alpha\beta}\phi_\alpha\phi_\beta}_{I_2}$$

~~Since~~

(10). $\Omega \in \Gamma_2^+$, i.e. $|H^2| - |L|^2 \geq 0$

Then $\mathcal{L}_2 = \oint H \phi_\nu^2 + \oint L_{\alpha\beta} \phi_\alpha \phi_\beta$

$\leq \oint H \phi_\nu^2 + \oint H |\nabla\phi|^2 \leq \oint H \qquad (\phi_\nu^2 + |\nabla\phi|^2 \leq 1)$

$\frac{1}{2}\mathcal{L}_1 = \oint_{\partial\Omega} \Delta\phi(\phi_\nu - 1)$

$= \oint_{\partial\Omega} (\bar{\Delta}\phi - H\phi_\nu)(\phi_\nu - 1) \qquad \left(\begin{matrix}\bar{\Delta}\phi \geq 0 \\ \phi_\nu - 1 \leq 0\end{matrix}\right)$

$\leq \oint H \phi_\nu (1 - \phi_\nu)$

$\leq \frac{1}{2} H$

$\mathcal{L}_2 \leq \oint H - \oint (H g_{\alpha\beta} - L_{\alpha\beta}) \phi_\alpha \phi_\beta$

$(H g_{\alpha\beta} - L_{\alpha\beta} = (T_1)_{\alpha\beta}(L_{\partial\Omega}) \geq 0, \quad \partial\Omega \in \Gamma_2^+)$

$\phi_\nu \leq (1 - |\nabla\phi|^2)^{\frac{1}{2}}, \quad (1-x)^{\frac{1}{2}} \leq 1 - \sum_{k=1}^{\infty} C_k x^k, \quad C_k > 0.$

$\oint (\Delta\phi)\phi_\nu \leq \oint (1 - \Sigma C_k |\nabla\phi|^{2k}) \Delta\phi$

$= -\sum_{k=1}^{\infty} C_k J_k$

$J_k = \oint \Delta\phi |\nabla\phi|^{2k} = -2k \oint \phi_{\alpha\beta} \phi_\alpha \phi_\beta |\nabla\phi|^{2(k-1)}$

Thus, $\mathcal{L}_1 \leq \frac{2}{3} \oint (H g_{\alpha\beta} - L_{\alpha\beta}) \phi_\alpha \phi_\beta$

Hence, $LHS \leq \oint_{\partial\Omega} H - \frac{1}{3} \underbrace{\oint_{\partial\Omega} (H g_{\alpha\beta} - L_{\alpha\beta}) \phi_\alpha \phi_\beta}_{\geq 0, \text{ when } \partial\Omega \in \Gamma_2^+;\ = 0 \text{ when } \Omega = B}$

See www.math.princeton.edu/~cheng for more details.