Some models for non-local Monge-Ampère equations

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Two Models of

The Monge Amp

realized as the i

formations of the

(det $D^2u = 1 \sim$ i

det $D^2u = f^n \sim$ inf

$\det a_{ij} = f^{n-1}$ –

minors of D^2u,

We may consider, then, a kernel K_I.
(we will stay with $K_I = |x|^{-(n+2s)}$, the Δ^s kernel, and look at the equation.

$$\mathcal{M}a^s(u)(x) = \inf_A \int \left[u(x+y) + u(x-y) - 2u(x)\right] |Ay|^{-(n+2s)}\, dy$$

where A ranges through the set Σ:
of positive matrices with determinant $\underline{\underline{1}}$.

(work with Fernando Charro).

The second model, the "Monster Monge-Ampere" was suggested by Luis Silvestre, and I will discuss joint work with him.

It consists, again in choosing a given kernel, (we again focus in $|X|^{-(n+2s)}) = K_0$ and consider all possible rearrangements K of K_0.

Then
$$MMA = \inf_K \int (u(x+y) - u(x) - \dots)\, K(y)\, dy$$

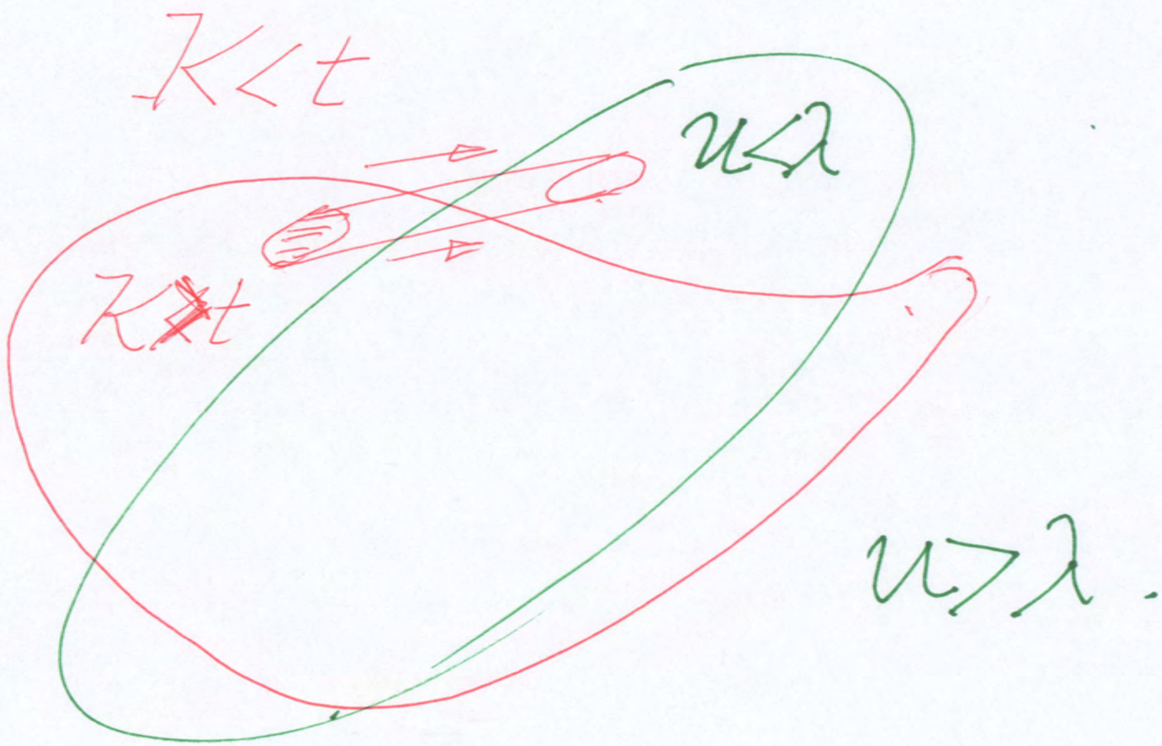
over all $K : |\{K > t\}| = |\{K_0 > t\}|$ for all $t > 0$.

[redacted]

This equation is well posed only for <u>convex functions</u>.

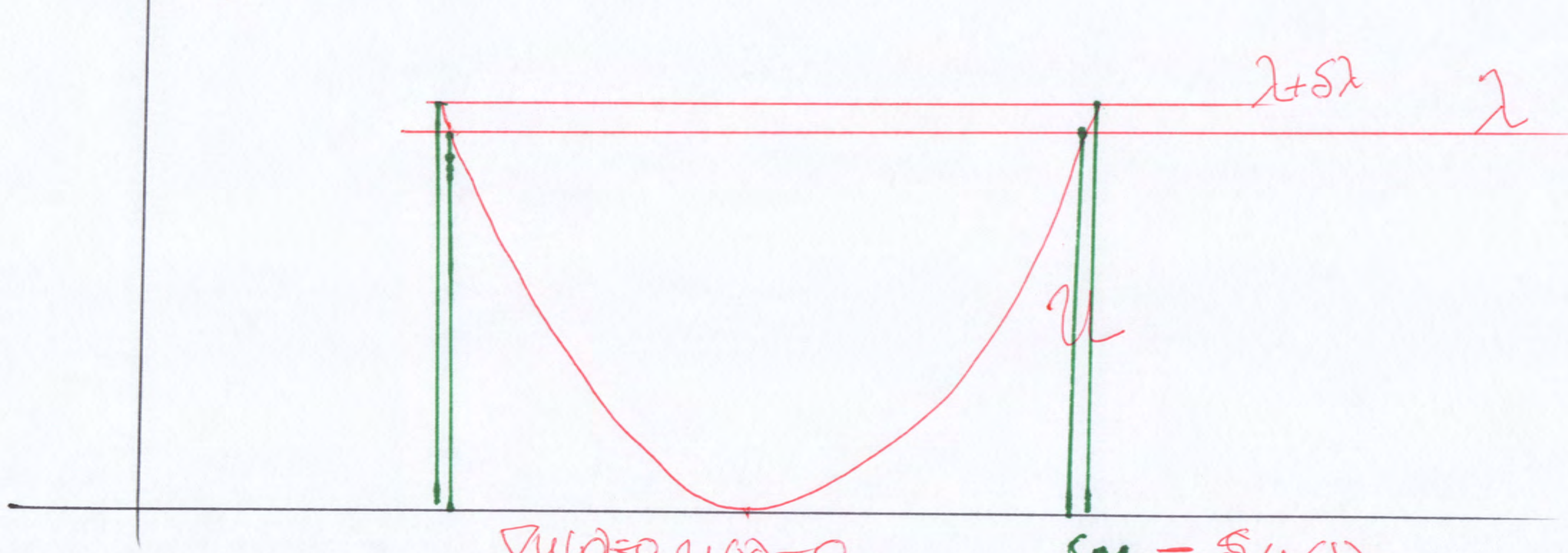
This apparently disorganized equation, has though a beatiful geometric description.
The first observation is that the level surfaces of K must coincide with those of u:

Assume that $|\{K > t\}| = |\{u < \lambda\}|$ and the sets do not coincide.



Let us find a formula for the operator in terms of u: We integrate in λ, the levels of u $(K(y)=t \quad = ?)$.

$$\int \Big[\overbrace{u(x+y)}^{\lambda} - \overbrace{u(x)}^{0} - \overbrace{\nabla u(x)\, y}^{0}\Big] \overbrace{K(y)}^{?} \overbrace{dy}^{\mu'(\lambda)\, d\lambda}.$$



But the "t" in $K(y)$, is the value that matches $|\{K>t\}|$ with $\mu(\lambda)$.

For Δ^s,

$$\left|\left\{|x|^{-n+2s}>t\right\}\right|=\left|\left\{|x|<t^{-1/n+2s}\right\}\right|=$$

$$\boxed{t^{-n/n+2s}}=\boxed{\mu(\lambda)}$$

$$mma=\int_0^\infty \lambda\,\mu(\lambda)^{-n+2s/n}\,d\mu(\lambda)=\boxed{\frac{2s}{n}\int_0^\infty \mu(\lambda)^{-\frac{2s}{n}}\,d\lambda}$$

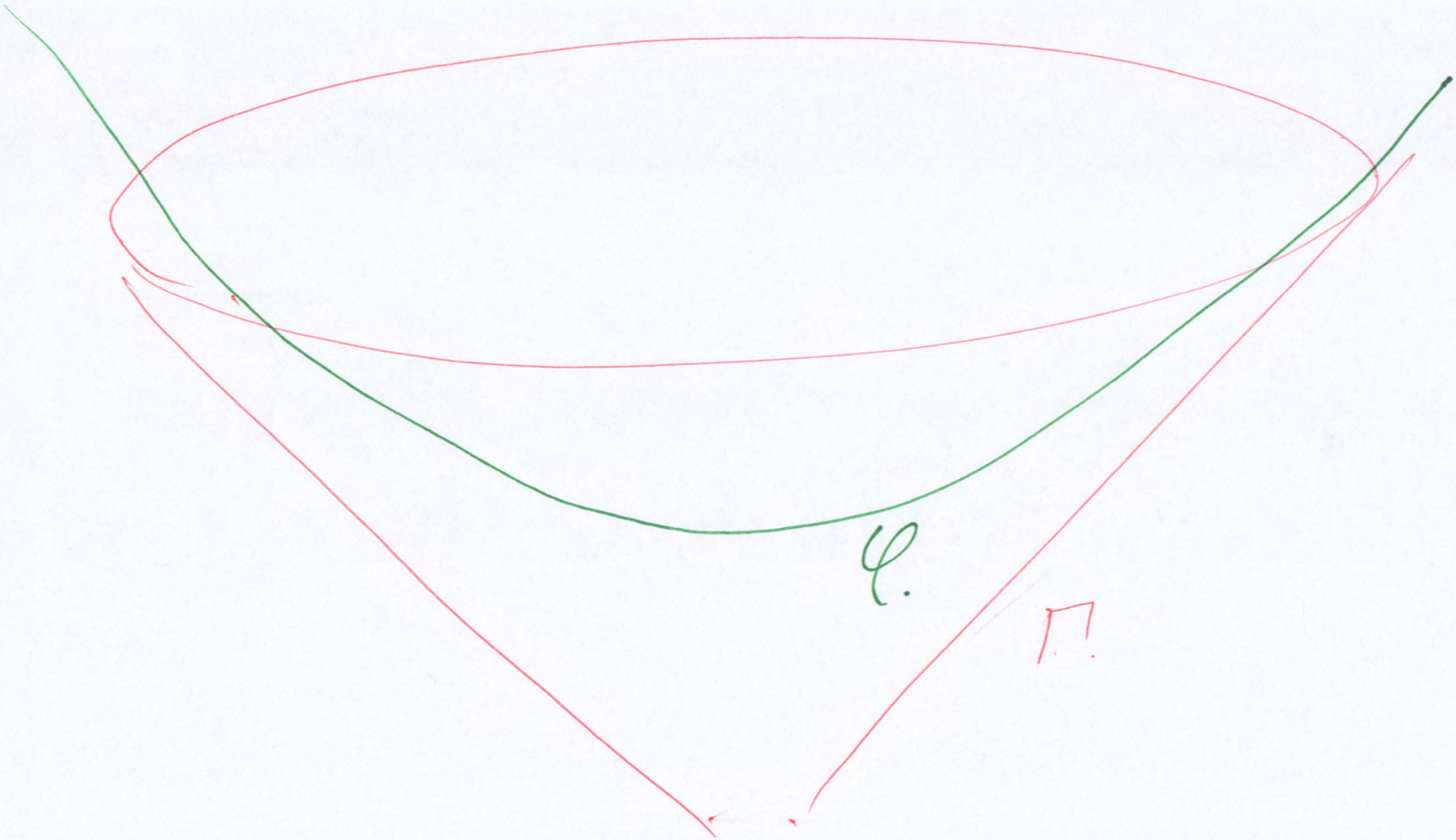
Existence, regularity

Simplest configuration.

= "Boundary values at ∞."

We prescribe φ:

- locally strictly convex and smooth; asymptotic to a smooth, strictly convex cone at infinity


C.
П.

Find a function $u(x)$ such that
$MA^s(u)$ (resp $MMA^s(u)$) $= u - \varphi$.

Simplest "test case":
- Schroedinger type equation ~
$$"\Delta u = u" + \dots$$
- The s of $\Delta^s > 1/2$ for integrability at infinity

$- \varphi$ with power decay to the cone,
to build a supersolution.

(Note that
φ is a subsolution:
$$Ma(\varphi) > 0 = \varphi - \varphi.$$
and $\varphi + C$ a supersolution for C large

$Ma(\varphi+C)$ ~~=~~ is bounded $(= Ma(\varphi)) \leq M$
and the RHS $= \varphi + C - \varphi = C \, (\geq) \, M$.

Affine model (with F. Charro).

- Build barrier.
- Solution is semiconcave.
- For any line, the fractional Lap.

$$\int \frac{[u(x+te)-u(x)]}{t^{\cancel{1}+2s}}\,dt \geq c_0 > 0.$$

- Uniform ellipticity: Given $M>0$, and K compact, $\exists$ a constant $\delta_0 > 0$ such that if $L_A u < M$, then $\delta_0 \mathrm{Id} \leq A \leq \frac{1}{\delta_0}\mathrm{Id}$.

- The solution u satisfies a uniformly elliptic equation with smooth kernels.

  Non local Evans Krylov applies.

- Symmetric functions of the Hessian?

- Truncated kernel: $|x|^{-(n+2s)}\eta(|x|)$?
  - Solutions with quadratic growth.
  - Jorgens-Calabi theorem?

MMA (with Silvestre)

As before, ℓ with same hypoteses.

Solve

$$u - \ell = MMA(u)(x) = \inf \int [u(x+y) - u(x) - \ell(y)]\, K(y)\, dy$$

over all K equidistributed to K_0.

- all supporting planes ℓ at x.

Main steps on the proof.

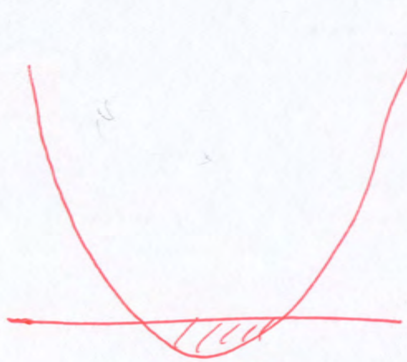
- Max of two subsolutions, u_1, u_2, is a subsolution?

(Observation, if l_1, l_2 are supporting planes, and $l_3 = t\, l_1 + (1-t)\, l_2$. then for any operator $L_K(u-l_3)$, we have

$$L_K(u-l_3) = t\, L_K(u-l_1) + (1-t)\, L_K(u-l_2)$$

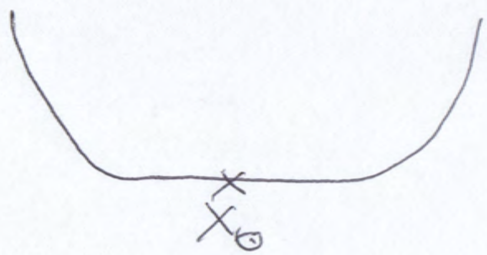
"To decide if u is a subsolution at X_0 it is enough to check it at the extremal points of the set of tangent slopes.

- A perturbation argument for a subsolution:
  - If $MMA\, u(x_0) \geq (u-\varphi) + \delta$, we can improve (we use the geometric description)
  - If u is strictly convex at the point just truncate:



In the flat parts MMA is constant and at the edge we use the previous observation.

If u has a flat part at the point



a) We may assume that x_0 is a strictly extremal point of $\{u = 0\}$.

Indeed, on the flat part, $MMA\,w = ct$, while the right hand side: $u - \varphi = -\varphi$ is concave. So its minimum is attained at an extremal point of $\{u = 0\}$.

We are now in condition to look at

$$w = \sup_{u \text{ subsol-}} u$$

We need to prove that w is a "subsolution".

In fact, we will start by proving that w is semi-concave.

For that purpose, we will consider incremental quotients $u(x+h) + u(x-h) - 2w(x)$ for u a subsolution.

We need to test the incremental quotient with an operator for which w is a supersolution.

We prove: a) there exists a plane ℓ for which $MMA_{\ell}\, w \leq w - \ell$.

(In the spirit of the perturbation arguments)

b) The sum of two subsolutions is a subsolution (with possibly different right hand sides).

Finally, the standard argument:

At a positive maximum, x_0, of

$$u(x_0+h) + u(x_0-h) - 2w(x_0) = M$$

we have that

i) $w(x_0) + M = \frac{1}{2}\left[u(x_0+h) + u(x_0-h)\right]$

ii) $w(x) + M \geq \frac{1}{2}\left[u(x+h) + u(x-h)\right]$

Therefore if L is the (linear) operator that corresponds to $MMA_\ell\, w(x_0)$,

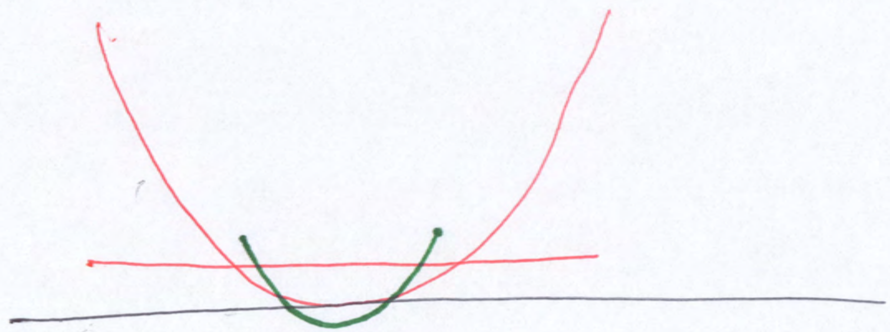
$$\begin{array}{lcl} L\, w(x_0) & \geq & \frac{1}{2}\left[L\,(u(x_0+h)) + L\,(u(x_0-h))\right], \\ \wedge\!| & & \vee\!| \\ (w-\varphi)(x_0) & & \frac{1}{2}\left[u-\varphi(x_0+h) + (u-\varphi)(x_0-h)\right]. \end{array}$$

or.

We pass to the limit in $u \to w$.

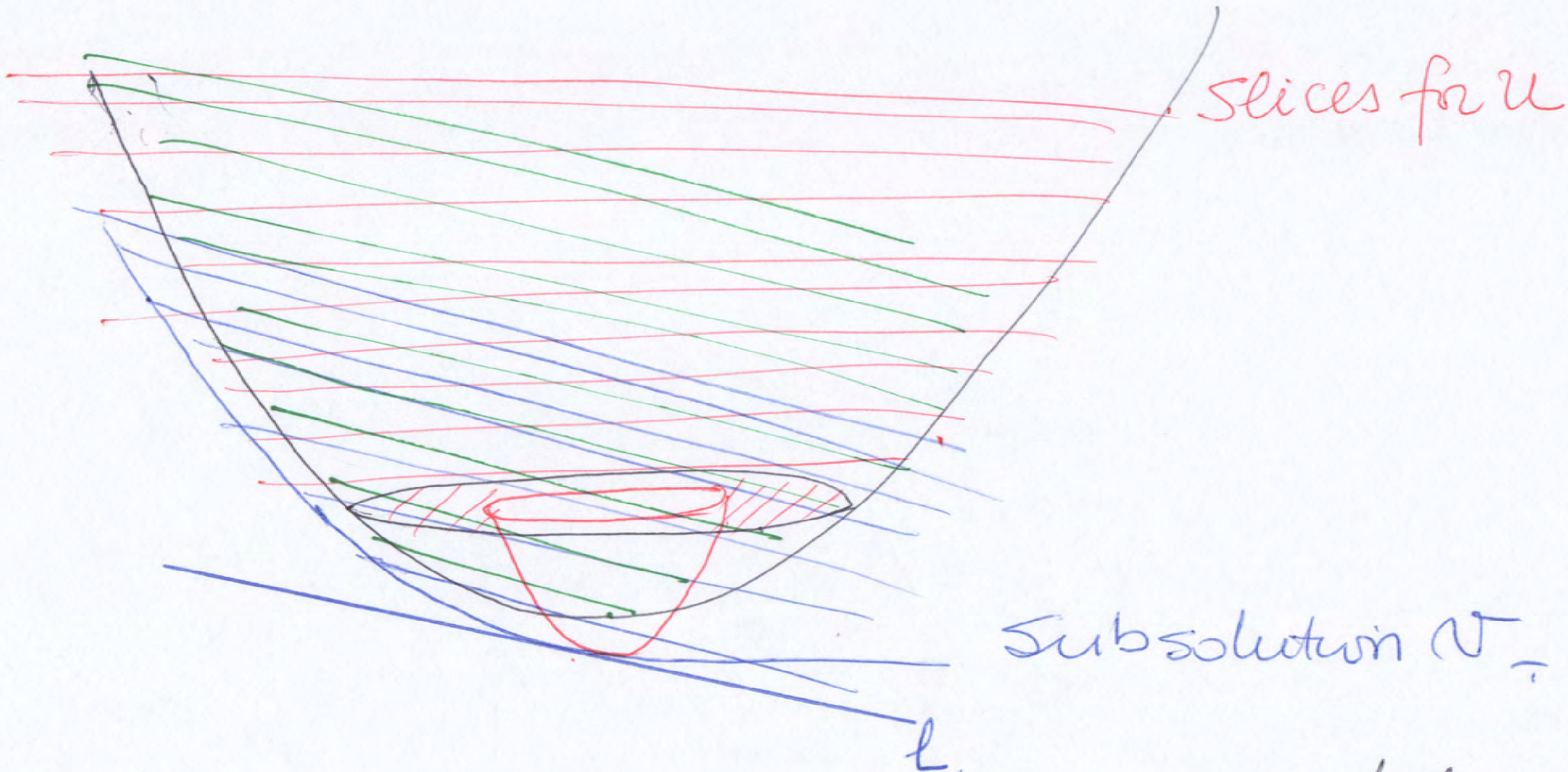
Idea of the proof: that u is a subsolut.

$$MMA_{\ell=0}(u)(0) < (u-\varphi) - \delta_0.$$



Since $MMA_{\ell=0}(u) < (u-\varphi) - \delta_0$, the

integral $\displaystyle\int \frac{1}{\mu^{2s/n}}\, d\lambda$ converges.



As λ_k goes to zero, the slope of ℓ goes to zero, as well as the contribution from the small paraboloid.

Jorgens-Calabi?

Pompeiu problem?

Thank you for your attention

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