① Q: Let μ be a Radon measure on $\mathbb{R}^m$, satisfyies

$\mu(B(x,r)) = r^n$, for $0 \le n \le m$.

$\forall r, \forall x \in \operatorname{spt}\mu = \{y \in \mathbb{R}^m, \mu(B(y,s)) > 0, \forall s > 0\} \triangleq \Sigma$.

If $n \in \mathbb{N}$, $m = n+1$.

$\Sigma = \begin{cases} \mathbb{R}^n \times \{0\}. \\ \text{if } r \geq 3 \quad \{x_4^2 = x_1^2 + x_1^2 + x_3^2\} \end{cases}$

Kowalski – Preiss

② History:

1. Besicovitch (1930's)

Q: μ Radon measure on $\mathbb{R}^2$.

$0 < \lim_{r \to 0} \frac{\mu(B(x,r))}{r} < \infty$, μ.a.e. $x \in \mathbb{R}^2$.

1-density exist.

A: μ is 1-rectifiable (i.e. supp $\subseteq$ C^1 curve in the set of ~)

2. Marstrand (1950's)

If $0 < \lim_{r \to 0} \frac{\mu(B(x,r))}{r^n} < \infty$. μ.a.e ~~$x \in \mathbb{R}^n$~~ on A. ($\mu A > 0$)

n. density exists.

$\Rightarrow n \in \mathbb{N}$

3. Preiss (80's) ~~If~~

If the n. density exists. $\Rightarrow$ "Σ is n ~~rect~~ rectifiable".

③. Preiss.

Let μ, ν Radon on $\mathbb{R}^n$, $B_r = B_r(0)$.

$$F_r(\mu,\nu) = \sup_f \left\{ \left| \int f\,d\mu - \int f\,d\nu \right| : \operatorname{Lip} f \le 1,\ \operatorname{spt} f \subseteq B_r,\ f \ge 0 \right\}$$

$$d(\mu,\nu) = \sum_{i=1}^{\infty} 2^{-i} \min\{1, F_i(\mu,\nu)\}.$$

④. Wasserstein distance $1 \le p < \infty$.

Let μ & ν probability measure on $\mathbb{R}^m$.

$$W_p(\mu,\nu) = \inf_\pi \left(\int_{\mathbb{R}^m \times \mathbb{R}^m} |x-y|^p \, d\pi(x,y) \right)^{\frac{1}{p}}.$$

where $\pi(A \times \mathbb{R}^m) = \mu(A)$

$\pi(\mathbb{R}^m \times B) = \nu(B)$

⑤. Kantorovitch duality.

$$W_1(\mu,\nu) = \sup_f \left\{ \left| \int f\,d\mu - \int f\,d\nu \right|,\ \operatorname{Lip} f \le 1 \right\}.$$

⑥. Tolsa

Regularity of Ahlfors regular measure

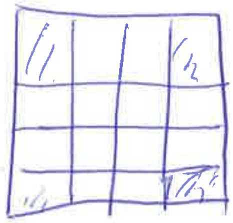
μ is n-ahlfors regular if

$\exists\, C > 1$, $\forall x \in \Sigma$, $\forall r > 0$.

$$C^{-1} r^n \le \mu(B(x,r)) \le C r^n.$$

Examples: 1. V n. dim plane in $\mathbb{R}^n$.

$$\mu = \mathcal{H}^n \llcorner V.$$

2.  4 corner Cantor set Σ.

$\underbrace{H^1 \llcorner \Sigma}_{\mu}$ 1-ahlfors regular

7. μ is n. uniformly rectifiable if.

$\exists\, \theta \in (0,1)$, $\exists\, L > 0$, given $x_0 \in \Sigma$, $r_0 > 0$.

$\exists\, g : B(0,r) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$. $g(0) = x_0$. g L Lips,

$$\mu(B(x_0,r_0) \cap g(B^n(0,r_0))) \geq \theta\, \mu(B(x_0,r_0)).$$

8. $\chi_{B_2} \leq \varphi \leq \chi_{B_3}$

$B = B(x,r)$. $\varphi_B(y) = \varphi\left(\frac{y-x}{r}\right)$ μ. n. ahlfors reg.

$$\tilde{\alpha}_p(x,r) = \tilde{\alpha}_p(B) = \frac{1}{r^{\frac{n+p}{p}}} \inf_{\substack{L \cap \Sigma \neq \emptyset \\ L \text{ n. plane}}} W_p\left(\varphi_B \mu ;\ c_{B,L}\, \varphi_B \mathcal{H}^n \llcorner L\right)$$

⑨. Tolsa

μ n. ahlfors regular, $1 \le p \le 2$

$\exists\, C_0 > 0$, s.t. $\displaystyle\int_{B(x_0,r_0)} \int_0^{r_0} \tilde{\alpha}_p(x,r)\, \frac{dr}{r}\, d\mu(x) \le C_0\, \mu(B(x_0,r_0))$.

$\forall x_0 \in \Sigma$, $\forall r_0 > 0$.

iff μ is n. unif. rect.

($\tilde{\alpha}_p(x,r) \frac{dr}{r} d\mu(x)$ is called a Carleson measure)

⑩. Joint J. Azzam & C. David.

μ is doubling if $\exists\, C > 0$ s.t. $\forall x \in \Sigma$, $\forall r > 0$.

$$\mu(B(x,2r)) \le C \mu(B(x,r))$$

$$\mu_{x,r}(A) = \frac{\mu(rA + x)}{\mu(B(x,r))}$$

Let $0 \le d \le m$. $d \in \mathbb{N}$,

$$\alpha_d(x,r) = \inf_{\substack{A \text{ affine} \\ d\text{-dim}}} W_1\left(\mu_{x,r}|_B,\ c_r \mathcal{H}^d(V|_B\right)$$

$$\alpha(x,r) = \min_{0 \le d \le m} \alpha_d(x,r)$$

(11). **Thm** (ADT).

If μ doubling and $\int_0^1 \alpha(x,r)\frac{dr}{r}$ is bounded. μ-a.e. x

Then $\exists\, S_0, \dots, S_n$. s.t $\mu(\Sigma \setminus \bigcup_{i=0}^{n} S_i) = 0$.

& S_d is d rectifiable,

& $\mu \llcorner S_d \ll \mathcal{H}^d \llcorner S_d \ll \mu \llcorner S_d$.

(END)