Spaces with Ricci curvature bounded from below

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Lessons

1) On the definition of spaces with Ricci curvature bounded from below

2) Analytic properties of $RCD(K, N)$ spaces

3) Geometric properties of $RCD(K, N)$ spaces

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On the definition of spaces with Ricci curvature bounded from below

- Introduction
- The gradient flow of the relative entropy w.r.t. W_2
- The gradient flow of the Dirichlet energy w.r.t. L^2
- The heat flow as gradient flow

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Completion / Compactification

A common practice in various fields of mathematic is to start studying a certain class of ‘smooth’ or ‘nice’ objects, and to close it w.r.t. some relevant topology.

In general, the study of the limit objects turns out to be useful to understand the properties of the original ones.

Gromov's plan

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Bounds from above/below on sectional curvature	Gromov-Hausdorff convergence	Alexandrov spaces
Bounds from below on the Ricci curvature	measured Gromov-Hausdorff convergence	$CD(K,N)$ spaces / $RCD(K,N)$ spaces

Aim of the game

(1) To understand what it means for a metric measure space to have Ricci curvature bounded from below

(2) To prove in the non-smooth setting ‘all’ the theorems valid for manifolds with $\mathrm{Ric} \geq K$, $\dim \leq N$ and their limits

(3) To better understand the geometry of smooth manifolds via the study of non-smooth objects

The curvature condition

Theorem (Sturm-VonRenesse '05) *- see also Otto-Villani and Cordero Erausquin-McCann-Schmuckenschlager*
Let M be a smooth Riemannian manifold. Then the following are equivalent:

i) The Ricci curvature of M is uniformly bounded from below by K

ii) The relative entropy functional is K-convex on the space $(\mathscr{P}_2(M), W_2)$

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Definition (Lott-Villani and Sturm '06) $(X, \mathsf{d}, \mathfrak{m})$ has Ricci curvature bounded from below by K if the relative entropy is K-convex on $(\mathscr{P}_2(X), W_2)$. Called $CD(K, \infty)$ spaces, in short.

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Basic features of the $CD(K, \infty)$ condition:

- Compatibility with the Riemannian case
- Stability w.r.t. mGH convergence
- More general $CD(K, N)$ spaces can be introduced

Finsler structures are included

Cordero-Erasquin, Villani, Sturm proved that $(\mathbb{R}^d, \| \cdot \|, \mathcal{L}^d)$ is a $CD(0, \infty)$ space (in fact $CD(0, d)$) for any norm.

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Some differences between the Finsler and Riemannian worlds:

Analysis:	Geometry:
Tangent / cotangent spaces can't be indentified	no Abresch-Gromoll inequality
No natural Dirichlet form	no Splitting theorem

Some observations

- For a given Finsler manifold the following are equivalent:
  - The manifold is Riemannian
  - The Sobolev space $W^{1,2}$ is Hilbert
  - The heat flow is linear

- The heat flow can be seen as:
  - Gradient flow of the Dirichlet energy w.r.t. L^2
  - Gradient flow of the relative entropy w.r.t. W_2

The idea

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- understand who is the heat flow on $CD(K,\infty)$ spaces
- show that such flow is stable w.r.t. mGH convergence.

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Plan pursued in:
G. '09
G., Kuwada, Ohta '10
Ambrosio, G., Savaré '11

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Definition of Gradient Flow: the smooth case

Let $(x_t) \subset \mathbb{R}^d$ a smooth curve and $f : \mathbb{R}^d \to \mathbb{R}$ a smooth functional. Then

$$
\begin{aligned}
f(x_0) - f(x_t) &\leq \int_0^t |x_s'||\nabla f|(x_s)\,\mathrm{d}s \\
&\leq \frac{1}{2}\int_0^t |x_s'|^2 + |\nabla f|^2(x_s)\,\mathrm{d}s.
\end{aligned}
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Therefore

$$x_t' = -\nabla f(x_t), \qquad \forall t \geq 0$$

$$\Updownarrow$$

$$f(x_0) = f(x_t) + \frac{1}{2}\int_0^t |x_s'|^2 + |\nabla f|^2(x_s)\,\mathrm{d}s, \quad \forall t > 0.$$

Definition of Gradient Flow: the metric setting

- $|\dot{x}_t| := \lim_{h \to 0} \dfrac{d(x_{t+h}, x_t)}{|h|}$ for an abs.cont. curve (x_t)
- $|\partial^- F|(x) := \overline{\lim_{y \to x}} \dfrac{(F(x) - F(y))^+}{d(x, y)}$
- The weak chain rule

$$F(x_0) \le F(x_t) + \frac{1}{2} \int_0^t |\dot{x}_s|^2 + |\partial^- F|^2(x_s)\, \mathrm{d}s, \qquad \forall t > 0.$$

holds for K-convex and l.s.c. $F : X \to \mathbb{R} \cup \{+\infty\}$.

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Definition (x_t) is a Gradient Flow for the K-conv. and l.s.c. functional F provided $(x_t) \subset \{F < \infty\}$ is a loc.abs.cont. curve and

$$F(x_0) = F(x_t) + \frac{1}{2}\int_0^t |\dot{x}_s|^2 + |\partial^- F|^2(x_s)\,\mathrm{d}s, \quad \forall t > 0.$$

General results about GF of K-convex functionals

Existence Granted if the space is compact and $F(x_0) < \infty$ (Ambrosio, G., Savaré '04 (after De Giorgi))

Uniqueness False in general

Basic facts about the GF of the Entropy

Thm. (G. '09)
Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $CD(K, \infty)$ space.

Then for $\mu \in \mathscr{P}_2(X)$ with $\mathrm{Ent}_{\mathfrak{m}}(\mu) < \infty$ the GF of $\mathrm{Ent}_{\mathfrak{m}}$ starting from μ exists and is unique.

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Furthermore, such flow is stable w.r.t. mGH-convergence of the base space.

mGH convergence of compact spaces

$(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ converges to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ in the mGH sense if there is (Y, d_Y) and isometric embeddings ι_n, ι_∞ of the X's into Y such that

$$(\iota_n)_\sharp \mathfrak{m}_n \qquad \text{weakly converges to} \qquad (\iota_\infty)_\sharp \mathfrak{m}_\infty$$

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We say that $n \mapsto \mu_n \in \mathscr{P}(X_n)$ weakly converges to $\mu_\infty \in \mathscr{P}(X_\infty)$ provided

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Γ-convergence of the entropies

Thm. (Lott-Sturm-Villani)
Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ be converging to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$. Then:

- $\Gamma - \underline{\lim}$ inequality: for every sequence $n \mapsto \mu_n \in \mathscr{P}(X_n)$ weakly converging to $\mu_\infty \mathscr{P}(X_\infty)$ we have

$$\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty) \leq \varliminf_{n\to\infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n).$$

- $\Gamma - \overline{\lim}$ inequality: for every $\mu_\infty \in \mathscr{P}(X_\infty)$ there is a sequence $n \mapsto \mu_n \in \mathscr{P}(X_n)$ weakly converging to μ_∞ such that

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Cor. The $CD(K, \infty)$ condition is closed w.r.t. mGH convergence.

$\Gamma - \underline{\lim}$ for the slopes

Thm. (G. '09) Let X_n be $CD(K,\infty)$ spaces mGH-converging to X_∞ and μ_n weakly converging to μ_∞. Then

$$|\partial^- \mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_\infty) \leq \varliminf_{n\to\infty} |\partial^- \mathrm{Ent}_{\mathfrak{m}_n}|(\mu_n).$$

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Cor. 1 Let X_n be mGH-converging to X_∞ and μ_n be weakly converging to μ_∞ be such that

$$\lim_{n\to\infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n) = \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty) < \infty.$$

Then the GF of $\mathrm{Ent}_{\mathfrak{m}_n}$ starting from μ_n converge to the GF of $\mathrm{Ent}_{\mathfrak{m}_\infty}$ starting from μ_∞.

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Then the GF of $\mathrm{Ent}_{\mathfrak{m}_n}$ starting from μ_n converge to the GF of $\mathrm{Ent}_{\mathfrak{m}_\infty}$ starting from μ_∞.

Cor. 2 The condition '$CD(K,\infty)$+linearity of the GF of the entropy' is closed w.r.t. mGH convergence.

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Variational definition of $|Df|$ on $\mathbb{R}^d$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be smooth.

Then $|Df|$ is the minimum continuous function G for which

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t$$

holds for any smooth curve γ

Test plans

Let $\pi \in \mathscr{P}(C([0,1],X))$. We say that π is a test plan provided:

- for some $C > 0$ it holds

$$\mathrm{e}_{t\sharp}\pi \leq C\mathfrak{m}, \qquad \forall t \in [0,1].$$

- it holds

$$\iint_0^1 |\dot{\gamma}_t|^2 \,\mathrm{d}t \,\mathrm{d}\pi < \infty$$

The Sobolev class $S^2(X, \mathsf{d}, \mathfrak{m})$

We say that $f : X \to \mathbb{R}$ belongs to $S^2(X, \mathsf{d}, \mathfrak{m})$ provided there exists $G \in L^2(X, \mathfrak{m})$, $G \geq 0$ such that

$$\int \left| f(\gamma_1) - f(\gamma_0) \right| \mathrm{d}\boldsymbol{\pi}(\gamma) \leq \iint_0^1 G(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma)$$

for any test plan $\boldsymbol{\pi}$.

Any such G is called weak upper gradient of f.

The minimal G in the $\mathfrak{m}$-a.e. sense will be denoted by $|Df|$

Basic properties

Lower semicontinuity From $f_n \to f$ $\mathfrak{m}$-a.e. with $f_n \in S^2$ and $|Df_n| \to G$ weakly in L^2 we deduce

$$f \in S^2, \qquad |Df| \le G$$

Locality

$$|Df| = |Dg| \qquad \mathfrak{m}\text{-}a.e.\ on\ \{f = g\}$$

Chain rule

$$|D(\varphi \circ f)| = |\varphi'| \circ f\,|Df|$$

for φ Lipschitz

'Leibniz rule'

$$|D(fg)| \le |f||Dg| + |g||Df|$$

for $f, g \in S^2 \cap L^\infty$

The Energy E and the Sobolev space $W^{1,2}$

We define $E : L^2(X, \mathfrak{m}) \to [0, +\infty]$ as

$$E(f) := \frac{1}{2} \int |Df|^2 \, d\mathfrak{m} \quad \text{if } f \in S^2(X), \qquad +\infty \quad \text{otherwise.}$$

Then E is convex and lower semicontinuous.

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The Sobolev space $W^{1,2}(X)$ is $W^{1,2}(X) := L^2(X) \cap S^2(X)$ endowed with the norm

$$\|f\|^2_{W^{1,2}} := \|f\|^2_{L^2} + \||Df|\|^2_{L^2}$$

$W^{1,2}(X)$ is a Banach space.

Laplacian (first definition)

We say that $f \in D(\Delta) \subset W^{1,2}(X)$ if $\partial^- E(f) \neq 0$.

In this case we define $\Delta f := -v$, where v is the element of minimal norm in $\partial^- E(f)$.

‘Integration by parts’

For $f \in D(\Delta)$ and $g \in W^{1,2}(X)$ we have

$$\left| \int g \Delta f \, \mathrm{d}\mathfrak{m} \right| \leq \int |Dg||Df| \, \mathrm{d}\mathfrak{m}.$$

For a C^1 map $u : \mathbb{R} \to \mathbb{R}$ we have

$$\int u(f) \Delta f \, \mathrm{d}\mathfrak{m} = - \int u'(f) |Df|^2 \, \mathrm{d}\mathfrak{m}.$$

Gradient flow of E w.r.t. L^2

For any $f_0 \in L^2(X, \mathfrak{m})$ there exists a unique map $t \mapsto f_t \in L^2(X, \mathfrak{m})$ such that

$$\frac{\mathrm{d}^+}{\mathrm{d}t} f_t = \Delta f_t, \qquad \forall t \geq 0.$$

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The result

Thm. (G., Kuwada, Ohta '10. Ambrosio, G. Savaré '11)
Let $(X, \mathsf{d}, \mathfrak{m})$ be a $CD(K, \infty)$ space and $\mu = f\mathfrak{m} \in \mathscr{P}_2(X)$ with $f \in L^2(X, \mathfrak{m})$. Let

- $t \mapsto f_t$ be the GF of E w.r.t. L^2
- $t \mapsto \mu_t$ be the GF of $\mathrm{Ent}_{\mathfrak{m}}$ w.r.t. W_2

Then

$$\mu_t = f_t\mathfrak{m} \qquad \forall t \geq 0.$$

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Then

$$\mu_t = f_t \mathfrak{m} \qquad \forall t \geq 0.$$

Idea of the proof: to show that $t \mapsto f_t \mathfrak{m}$ is a GF of $\mathrm{Ent}_{\mathfrak{m}}$ w.r.t. W_2 and conclude by uniqueness of the latter.

Spaces with Riemannian Ricci curvature bounded from below

$$\begin{aligned} RCD(K,N) &:= CD(K,N) + \text{linearity of the heat flow} \\ &= CD(K,N) + W^{1,2} \text{ is Hilbert} \end{aligned}$$

Thank you

Basic properties of the GF of E

Mass preservation $\int f_t \, \mathrm{d}\mathfrak{m} = \int f_0 \, \mathrm{d}\mathfrak{m}$ for every $t \geq 0$

Weak maximum principle If $f_0 \leq C$ then $f_t \leq C$ for every $t \geq 0$

Entropy dissipation For $0 < c \leq f_0 \leq C$ the map $t \mapsto \int f_t \log f_t \, \mathrm{d}\mathfrak{m}$ is absolutely continuous and it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f_t \log f_t \, \mathrm{d}\mathfrak{m} = -\int \frac{|Df_t|^2}{f_t} \, \mathrm{d}\mathfrak{m}$$

A non-trivial property of the heat flow - Kuwada's lemma

Suppose that $\mu_0 := f\mathfrak{m}$ is in $\mathscr{P}_2(X)$.

A non-trivial property of the heat flow - Kuwada's lemma

Suppose that $\mu_0 := f\mathfrak{m}$ is in $\mathscr{P}_2(X)$.

Then $\mu_t := f_t\mathfrak{m}$ is in $\mathscr{P}_2(X)$ and the curve $t \mapsto \mu_t$ is absolutely continuous w.r.t. W_2 and

$$|\dot{\mu}_t|^2 \le \int \frac{|Df_t|^2}{f_t} \, \mathrm{d}\mathfrak{m}$$