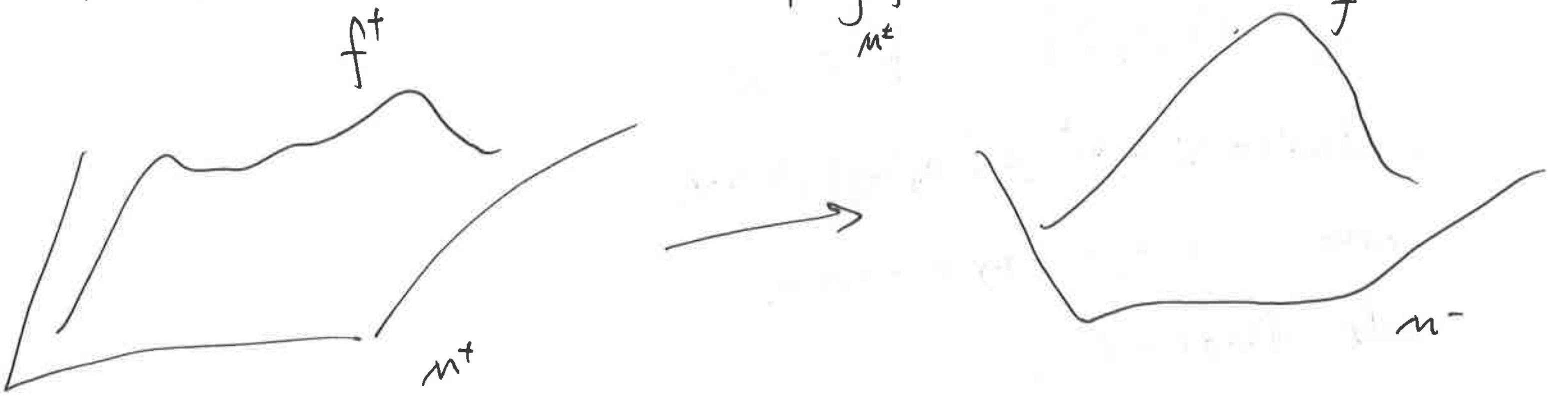
$M^{\pm} \subseteq \mathbb{R}^n$ $\qquad 1 = \int_{M^{\pm}} f^{\pm}\, dx$




$c \in C(\bar N)$, $N = M^+ \times M^-$

$\text{cost}(\gamma) = \int_{M^+ \times M^-} c(x,y)\, d\gamma(x,y)$. $\qquad d\gamma = g(x,y)\, dx\, dy$.

~~where~~ $\bar g(x,y) \in L^1(N)$

$\Gamma = \Gamma(f^+, f^-)$ $\qquad \Gamma^{\bar g} = \left\{ 0 \le g \le \bar g : \begin{array}{l} f^+(x) = \int_{M^-} g(x,y)\,dy \\ f^-(y) = \int_{M^+} g(x,y)\,dx \end{array} \right\}$ a.e. $(x,y)$.

$\min_{\gamma \in \Gamma^{\bar g}} \text{cost}(\gamma)$

$\bar g(x,y)$ = compacity constrain # between $x$ and $y$

$\bar g = +\infty \Rightarrow$ old problem (Kantorovich)

Kellerer '85 $\qquad \Gamma^{\bar g} \neq \emptyset \iff \forall\, U \times V \subseteq M^+ \times M^-$

Levin '84 $\qquad \int_U f^+ + \int_V f^- \le 1 + \int_{U \times V} \bar g$

$f^+(x) f^-(y) \notin \Gamma^{\bar g}$

Levin's Duality:

$S(x,y) = -c(x,y)$.

$$-\min_{\gamma \in \Gamma^{\bar g}} \text{cost}(\gamma) = \max_{g \in \Gamma^{\bar g}} \int S g = \inf_{\substack{\text{cts bdd } u,v,w \\ w \ge 0 \\ S(x,y) \le u(x) + v(y) + w(x,y)}} \int_{M^+} u f^+ + \int_{M^-} v f^- + \int_N w \bar g$$

e.g. fix $n=1$.

$$\mu^{\pm} = [-\tfrac{1}{2}, \tfrac{1}{2}], \qquad f^{\pm} = 1_{\mu^{\pm}}.$$

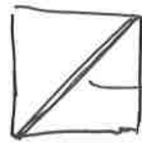
$$c(x,y) = \tfrac{1}{2}|x-y|^2 = \tfrac{1}{2}|x|^2 + \tfrac{1}{2}|y|^2 - x\cdot y.$$

equiv. $c(x,y) = -x\cdot y = -s(x,y)$.

take $\bar{g}(x,y) = C$.

e.g.1. if $\bar{g}(x,y) = 1$, $\Rightarrow \Gamma^{\bar{g}} = \{1_N\}$.

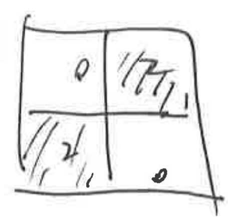
e.g.2. if $\bar{g}(x,y) = \infty$, $\Rightarrow$



all the mass are on the diagnal.

e.g.3. if $\bar{g}(x,y) = 2$.

$\Rightarrow$



proof of e.g.3. $\Rightarrow$ $u = v = 0$, $w(x,y) = [x\cdot y]_+$.

$g \in \Gamma^{\bar{g}}$ $\Rightarrow$ $s(x,y) \le u(x) + v(y) + w(x,y)$

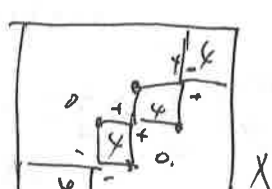
$(u,v,w) \in \mathcal{F}$.

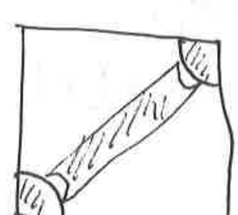
$\Rightarrow \int s g \le \int u f^+ + \int v f^- + \int \bar{g} w$

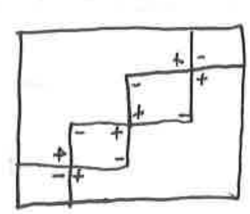
take sup of all g. and take inf of u, v, w.

then $\max \int s g = \inf_{(u,v,w)\in\mathcal{F}} \int u f^+ + \int v f^- + \int w \bar{g}$.

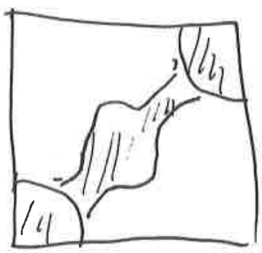
e.g.4. $\bar{g}(x,y) = 4$



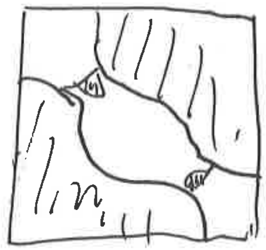




e.g. $\bar{g}(x,y) = 3$.



$\bar{g}(x,y) = 1.5$



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$g$ is extremal of $\Gamma^{\bar{g}}$ unless $g$ is the midpoint of a segment in $\Gamma^{\bar{g}}$.

Prop: $g \in \Gamma^{\bar{g}}$ is extremal $\iff$ $g = \begin{cases} \bar{g} & \text{on } W \subseteq N \\ 0 & \text{elsewhere} \end{cases}$ $= \bar{g} 1_W$

LEMMA: Fix $\Delta = (1, 1, \dots, 1)$ on $\mathbb{R}^n$ and $U \subseteq \mathbb{R}^{2n}$ with full Lesbegue density at $(x_0, y_0)$.

$H = [-\frac{1}{2}, \frac{1}{2}]^{2n}$

$\Rightarrow \forall \delta > 0 \ \exists V, \quad V \subseteq \delta H, \ |V| > 0.$

$V + (x_0, y_0) \pm \delta(\Delta, \Delta) \subseteq U.$

$V + (x_0, y_0) \pm \delta(\Delta, -\Delta) \subseteq U.$

proof of Prop

"$\Leftarrow$" easy: $g = \frac{g_+ + g_-}{2} \Rightarrow g_\pm = g$ a.e..

"$\Rightarrow$" suppose $g \neq \bar{g} 1_W$. for any $W \subseteq N$.

Then $U_\varepsilon = \{ 0 + \varepsilon < g < \bar{g} - \varepsilon \}$ has positive measure

Let $(x_0, y_0)$ be a point of full density

perturb $g$ by $g_\delta(x,y) = 1_{V + (x_0,y_0) + \delta(\Delta,\Delta)} + 1_{V+(x_0,y_0) - \delta(\Delta,\Delta)}$

$- 1_{V+(x_0,y_0) - \delta(\Delta,-\Delta)} - 1_{V+(x_0,y_0)+\delta(0,-\Delta)}$.

$g_\pm = g \pm \varepsilon g_\delta$.

Thm (Every optimizer is extreme)

Let $c \in (C \cap L^\infty)(N) \cap C^2(N - Z)$ $|Z| = 0$, $Z$ closed.

$\det D^2_{x,y} c(x,y) \neq 0$, on $N - Z$.

$0 \leq f^\pm \in L^1(M^\pm)$ $\bar{g} \in L^1(N)$. $\arg\min_{\Gamma^{\bar{g}}} \mathrm{cost}(g) \subseteq \mathrm{extreme}\ \Gamma^{\bar{g}}$

COR (Uniqueness under some hypothesis).

Proof If both $g_\pm \in \Gamma^{\bar{g}}$ minimizer.

So does $g = \frac{g_+ + g_-}{2}$.

Then Thm $\Rightarrow g$ extreme $\Rightarrow g_+ = g_-$

~~Proof Sup~~

proof. suppose $g \notin I_{W,\bar{g}}$, for any $W$.

a linear change variable $\tilde{y} = Ay$.

$$D^2_{x,\tilde{y}} c(x,\tilde{y}) = -\delta_{ij}.$$

perturbation $\tilde{g} = g + \varepsilon g_\delta$. lower the cost

~~achieve the cost~~

So $g$ is not optimal.

LEMMA. $M^\pm \subseteq \mathbb{R}^n$, $|M^\pm| = 1$, $M^\pm = -M^\pm$.

$f^\pm = 1_{M^\pm}$, $c(x,y) = -x\cdot y$, $\bar{g} = \bar{g}_p = p \cdot 1_{M^+ \times M^-}$.

$\frac{1}{p} + \frac{1}{q} = 1$, $W \subseteq N$, $\tilde{W} = N - W$. $R(W) = \{(x,-y) \mid (x,y) \in W\}$.

$$p 1_W \in \underset{\Gamma^{\bar{g}_p}}{\operatorname{argmin}} \operatorname{cost}(g) \iff q\, 1_{R(\tilde{W})} \in \underset{\Gamma^{\bar{g}_q}}{\operatorname{argmin}} \operatorname{cost}(g).$$

COR. for $p = q = 2$, then optimizer $2 1_W = 2 \cdot 1_{R(\tilde{W})}$.

Thm. $M^\pm \subseteq \mathbb{R}^n$, $|M^\pm| = 1$. $s \in C'(M^+, M^-)$ $f^\pm > 0$.

$\bar{g}$ bdd above and below, continuous, strictly possitive.

$\exists\, \eta > 1$, s.t. $\Gamma^{\bar{g}/\eta} \neq \phi$

$\Rightarrow$ Levin's inf attained by signed measure $\mu, \nu$ and $w = [s - u - v]_+$ of finite total variation.

proof. $(u,v,w) \in \mathcal{F} \Rightarrow (u, v, [s-u-v]_+) \in \mathcal{F}$. and better.

$$\inf_{u,v} \quad \overline{uf^+} + \overline{vf^-} + \int [s-u-v]_+ \bar{g}.$$

replace $(u,v)$ by $(u+\lambda, v-\lambda)$ ensures

$$\overline{uf^+} = \overline{vf^-}$$

use " coercivity ". of $[\quad]_+$.

to show any minimizing sequence $(u_k, v_k)$ has a priori $\|u_k\|_{L^1}$, $\|v_k\|_{L^1}$ bounds + pass to limits.

depending on $\eta > 1$ and bd for $f^\pm$.

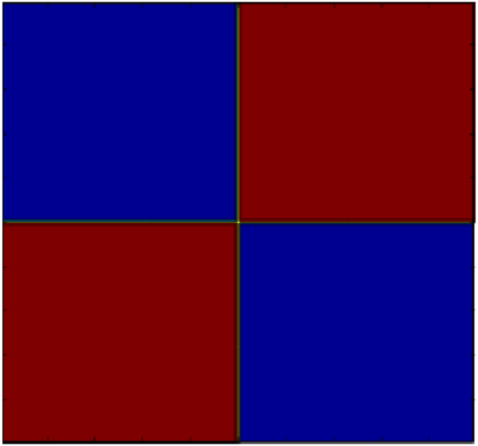


Figure : The two-by-two checkerboard solves $\bar{g} = 2$

$$g = \bar{g} \text{ in red regions; schematically } g(x,y) = g_2(x,y) = \begin{array}{|c|c|} \hline 0 & 2 \\ \hline 2 & 0 \\ \hline \end{array}$$

# Unexpected symmetries

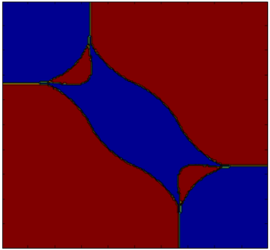
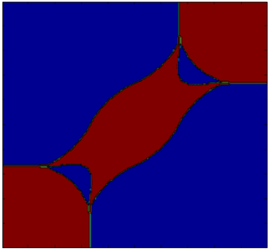


Figure : $g = \bar{g}$ in red regions solves $\bar{g} = 3/2$ above, $\bar{g} = 3$ below



# Numerics (with help from B Wetton)

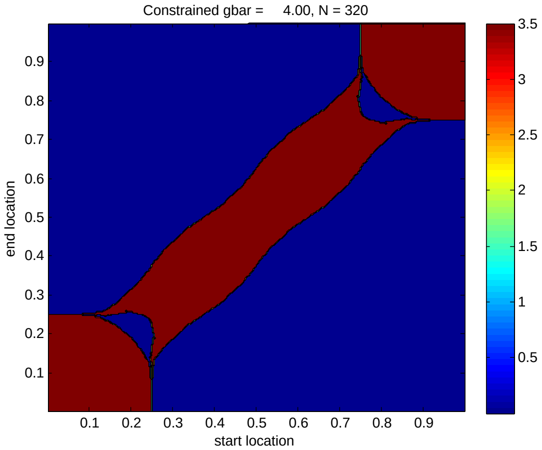



Figure : $g = \bar{g}$ in red regions solves $\bar{g} = 4$.