Regularity in optimal transportation

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This talk consists of three parts:

1. Review on the regularity under the strong A3 condition (Ma-Trudinger-Wang, Loeper).
2. Regularity under the weak A3 condition (Figalli-Kim-McCann, Wang)
3. Regularity in Monge's problem (Li-Santambrogio-Wang)

- I will not discuss in details the fundamental work of Caffarelli, it is well known now.
- I will not discuss the regularity on manifolds, which was studied by many people (Ph. Delanoe, A. Figalli, G. Loeper, Y.H. Kim-R. McCann, L. Rifford, C. Villani)

1. Optimal transportation

Let Ω and Ω^* be bounded domains in R^n.
Let f and g be mass densities on Ω and Ω^* satisfying

- $$0 \leq f \in L^1(\Omega),\ 0 \leq g \in L^1(\Omega^*),$$

$$\int_\Omega f = \int_{\Omega^*} g.$$

- $\exists$ constants $f_0, f_1, g_0, g_1 > 0$ such that

$$f_0 \leq f \leq f_1, \qquad g_0 \leq g \leq g_1.$$

Let (u, v) be the potential functions.

The optimal mapping T_u is given by

$$Du(x) = D_x c(x, T_u(x)) \tag{1}$$

Differentiate the above formula,

$$D^2 u(x) = D_x^2 c(x, T_u(x)) + D_{xy}^2 c(x, T_u(x)) DT.$$

We obtain the equation

$$\det[D_x^2 c(x, T_u(x)) - D^2 u(x)] = |D_{xy}^2 c| \cdot \frac{f(x)}{g(T_u(x))}. \tag{2}$$

The boundary condition:

$$T_u(\Omega) = \Omega^*. \tag{3}$$

- If

$$c(x, y) = x \cdot y,$$

we have the standard Monge-Ampere equation

$$\det D^2 u = f(x, u, Du) \quad \text{in } \Omega, \tag{4}$$

subject to the boundary condition:

$$Du(\Omega) = \Omega^* \tag{5}$$

- For Monge's cost

$$c(x, y) = |x - y|$$

$D_x^2 c$ is singular and $D_{xy}^2 c = 0$. The equation (2) has no meaning and so a different treatment is needed.

Regularity of the standard Monge-Ampere equation has been studied by many people, including
Calabi, Pogorelov, Nirenberg, Cheng-Yau, Caffarelli.

- Interior 2nd derivative estimate of Pogorelov.
- Higher regularity by Calabi.
- Higher regularity also follows from Evans-Krylov's theory.
- Minkowski problem by Pogorelov and Nirenberg in 2-dim. by Pogorelov and Cheng-Yau in high dim.
- Bernstein Thm by Jorgens, Calabi, Pogorelov, Cheng-Yau.
- $C^{2,\alpha}$ and $W^{2,p}$ by Caffarelli
  (2-dim by Heinz, and Nikolaev-Shefel, resp).

As a result, Caffarelli obtained the regularity of optimal mappings for the cost function

$$c(x, y) = |x - y|^2$$

(a) If $f, g > 0, \in C^{\alpha}$ and Ω^* is convex, then $u \in C^{2,\alpha}(\Omega)$

(b) If $f, g > 0, \in C^{0}$ and Ω^* is convex, then $u \in W^{2,p}_{loc}(\Omega)$ $\forall\ p > 1$ (the continuity is needed for large p).

(c) If $f, g > 0, \in C^{\alpha}$, both Ω and Ω^* are uniformly convex and $C^{2,\alpha}$, then $u \in C^{2,\alpha}(\bar{\Omega})$.

- If $f, g \in C^{1,1}$, $\partial\Omega, \partial\Omega^* \in C^{3,1}$, the global $C^{3,\alpha}$ regularity was obtained by P. Delanoe (dim 2) and J. Urbas (all dim).
- If $c_0 < f < c_1$, then $u \in W^{2,p}_{loc}$ for some $p > 1$ by De Philippis-Figalli-Savin, and Schmidt.

Caffarelli and Villani proposed to study the regularity of potential functions for more general cost functions.

- Caffarelli (ICM2002): Geometry of optimal transportation (local property of the potential function).
- Villani (Book2003): Regularity of the optimal transportation.
- These two problems are closely related.

We need to study the regularity for

$$\det[D_x^2 c(x, T_u(x)) - D^2 u(x)] = |D_{xy}^2 c| \cdot \frac{f(x)}{g(T_u(x))}. \qquad (*)$$

Theorem 1 (Ma-Trudinger-Wang).
The potential function u is C^3 smooth if the cost function c is smooth, f, g are positive, $f \in C^2(\Omega), g \in C^2(\Omega^*)$, and

A1 $\forall\, x, \xi \in R^n,\ \ \exists_1\, y \in R^n$ s.t. $\xi = D_x c(x, y)$ (for existence)

A2 $\left|D^2_{xy} c\right| \neq 0.$

A3 $\exists\, c_0 > 0$ s.t. $\forall\, \xi, \eta \in R^n,\ \xi \perp \eta$

$$\sum (c_{ij,rs} - c^{p,q} c_{ij,p} c_{q,rs}) c^{r,k} c^{s,l} \xi_i \xi_j \eta_k \eta_l \geq c_0 |\xi|^2 |\eta|^2,$$

where the subscripts of c before the comma mean derivatives in x, after the comma mean derivatives in y, and $c^{i,j}$ is the inverse of the matrix $c_{i,j}$.

B1 Ω^* is c-convex w.r.t. Ω
(namely $\forall\, x_0 \in \Omega$, $\Omega^*_{x_0} := D_x c(x_0, \Omega^*)$ is convex)

Remarks on the conditions

- A1 is for the existence of optimal mappings.
- A2 is natural for regularity.
- A3 is a structural condition, equivalent to

$$D^2_{p_k p_l} A_{ij}(x,p)\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2 \quad \forall\, \xi \perp \eta.$$

where $A(x, Du) = D^2_x c(x, T_u(x))$.

M-T-W: If Ω^* is not c-convex w.r.t. Ω (namely if B1 is violated), then $\exists\, f, g > 0, \in C^2$, such that $u \notin C^1$.

Loeper: If the structural condition A3 is violated (details shown later), then $\exists\, f, g > 0, \in C^2$, such that $u \notin C^1$.

Proof of Theorem 1:

- A priori estimates $+$ understanding the convexity of potentials and domains wrt cost function.
- The idea is similar to that in my early paper on the reflector problem (Inverse Problem 1996).

Geometric property of (A3) by Loeper

Let y_0, y_1 be two points in Ω^*. Let

$$\overline{y_0 y_1} = \{y_t : \quad c_x(x_0, y_t) = p_t, t \in [0,1]\}$$

be the c-segment relative to a point $x_0 \in \Omega$, connecting y_0, y_1, where $p_t = tp_1 + (1-t)p_0$, and $p_0 = c_x(x_0, y_0)$, $p_1 = c_x(x_0, y_1)$. Let

$$h_t(x) = c(x, y_t) - a_t,$$

where $t \in (0,1)$ and a_t are constants such that

$$h_t(x_0) = h_0(x_0) \quad \forall \quad t \in [0,1].$$

Then for $x \neq x_0$, near x_0, and $0 < t < 1$, we have the inequality

$$h_t(x) > \min\{h_0(x), h_1(x)\}$$

Further regularity

$$\det[D_x^2 c(x, T_u(x)) - D^2 u(x)] = |D_{xy}^2 c| \cdot \frac{f(x)}{g(T_u(x))}.$$

Theorem 2 (Liu-Trudinger-Wang): Suppose $c_0 < f, g < c_1$ and Ω^* is c-convex. Then we have the estimate

$$|D^2 u(x) - D^2 u(y)| \leq C\Big[d + \int_0^d \frac{\omega_f(r)}{r} dr + \int_d^1 \frac{\omega_f(r)}{r^2}\Big]$$

where $d = |x - y|$, ω_f is the oscillation of f.

- If f is Dini, then $u \in C^2$.
- If $f, g \in C^\alpha$, then $u \in C^{2,\alpha}(\Omega)$.

Proof.

The proof uses a perturbation argument and the regularity of Ma-Trudinger-Wang (Theorem 1). We also need

- Strict c-convexity of u (Trudinger-Wang 2009)
- Interior second derivative estimate of Pogorelov type for cost functions under A3w. Assume $u = 0$ on $\partial\Omega$, then

$$(-u)^p |D^2 u| \leq C \text{ in } \Omega$$

for some $p > 1$. (under some technical conditions on c which is satisfied in the perturbation argument)

Theorem 3 (Liu-Trudinger-Wang): If $f, g \in C^0$ and $f, g > 0$, then $u \in W^{2,p}_{loc}(\Omega)$

Proof.

- Normalization
  (another step towards the understanding of the geometry).
  $\forall\ x_0 \in \Omega$, let $y_0 = T(x_0)$ and let

$$c(x,y) \to [c(x,y) - c(x,y_0)] - [c(x_0,y) - c(x_0,y_0)],$$
$$u(x) \to [u(x) - u(x_0)] - [c(x,y_0) - c(x_0,y_0)],$$
$$v(y) \to [v(y) - v(y_0)] - [c(x_0,y) - c(x_0,y_0)],$$

  Then
$$c(x,y_0) \equiv 0, \quad c(x_0,y) \equiv 0$$
$$u(x_0) = 0, \quad Du(x_0) = 0$$
$$v(y_0) = 0, \quad Dv(y_0) = 0$$

- Now we make the transform

$$x \to D_y c(x, y_0), \quad y \to D_x c(x_0, y)$$

Then the level set of

$$S_h^0(x_0) = \{u < h\}$$

is convex (by the geometric property of A3),

$$c(x, y) = c \cdot y + (c_{ij,kl} - c_{ij,m} c_{m,kl}) x_i x_j y_k y_l + h.o.t.$$

Moreover in the level set we have

$$D_x^2 c(x, T_u(x)) \to 0 \quad \text{as } h \to 0,$$

In particular

$$\det[D_x^2 c(x, T_u(x)) - D^2 u(x)] \to \det[-D^2 u(x)]$$

- Asymptotic analysis

$$|S_h^0(x_0)| \approx h^{n/2}$$

Moreover, as $h \longrightarrow 0$,

$$c(x, y) \longrightarrow x \cdot y$$

$$u(x) \longrightarrow \frac{1}{2}|x|^2$$

in the limit of normalization.

- With the above local properties, we then use Caffarelli's method for the $W^{2,p}$ estimate.

Cost functions satisfying A3

$$c(x,y) = \begin{cases} -\log|x-y|, \\ \sqrt{1+|x-y|^2}, \\ \sqrt{1-|x-y|^2}, \\ |x-y|^2 \ \text{ on sphere}, \\ |x-y|^2 + |f(x)-g(y)|^2, \ f,g \text{ unif. convex}, |Df|,|Dg| < 1, \\ |x-y|^p, \ \ -2 < p < 1 \ p \neq 0. \end{cases}$$

- Cost functions not satisfying A3

$$c(x,y) = |x-y|^p, \quad p < -2 \text{ or } p > 1.$$

- If $c(x,y)$ satisfies A3, then $-c(x,y)$ does not.

Further remarks on the condition A3

Recall the assumption A3:

$$D^2_{p_k p_l} A_{ij}(x,p)\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2 \quad \forall\, \xi \perp \eta.$$

where $A(x, Du) = D^2_x c(x, T_u(x))$.

Loeper: If the structural condition A3 is violated, namely if $\exists\, \xi, \eta$ such that

$$D^2_{p_k p_l} A_{ij}(x,p)\xi_i\xi_j\eta_k\eta_l < 0$$

then $\exists\, f, g > 0, \in C^2$, such that $u \notin C^1$.

- To prove the result, he observed a geometric property of the condition (A3), and used an idea in [Ma-Trudinger-Wang].
- The idea is that construct a sequence of mass densities g_k which converges to the Dirac measure $\delta_{y_0} + \delta_{y_1}$ for proper y_0, y_1.

Regularity under (A3w)

M-T-W: Regularity if c satisfies (A3):

$$D^2_{p_k p_l} A_{ij}(x,p)\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2 \quad \forall\, \xi \perp \eta.$$

Loeper: Counterexample if $\exists\, \xi, \eta$ $(\xi \perp \eta)$ such that

$$D^2_{p_k p_l} A_{ij}(x,p)\xi_i\xi_j\eta_k\eta_l < 0$$

Question— (closing the gap between M-T-W and Loeper)
Regularity under the weak A3:

(A3w) $$D^2_{p_k p_l} A_{ij}(x,p)\xi_i\xi_j\eta_k\eta_l \geq 0 \quad \forall\, \xi \perp \eta?$$

Answer: Yes. We can establish

i). Strict c-convexity and $C^{1,\alpha}$ of potential functions

ii). $C^{2,\alpha}$ and $W^{2,p}$ estimates.

- Parts i) was obtained by Figalli-Kim-McCann assuming that the target domain is unif convex. The unif convexity can be removed.
- Part ii) follows if i) is proved.

Regularity under (A3w)

Theorem Assume that

- Ω, Ω^* bounded domains in R^n,
- $\Omega \subset \widetilde{\Omega}$, and $\widetilde{\Omega}, \Omega^*$ are c-convex wrt each other,
- $f_0 < f < f_1$, $g_0 < g < g_1$ for const f_0, f_1, g_0, g_1,
- $c \in C^\infty$, satisfying (A1), (A2) and (A3w).

Then

- The potential function u is strictly c-convex in Ω.
- $u \in C^{1,\alpha}(\Omega)$

Proof. The proof is similar to that of Caffarelli for cost $|x-y|^2$ and that of Figalli-Kim-McCann for general costs.

- Introduce c-cone $\vee_p^{D,h}$ (cone relative to cost function c), where p is the vertex, D is the base, and h is the height.
- Estimate size of sub-gradient $\partial_- \vee(p)$ of the c-cone $\vee$. Here the sub-gradient of a function is

$$\partial_-\varphi(x_0) = \{p \in R^n \mid \ \varphi(x) \geq \varphi(x_0)+p\cdot(x-x_0)-o(|x-x_0|)\}.$$

We show that, after the normalization as above, and if $\mathrm{diam} D \leq \delta_0$,

$$|\partial_- \vee(p)| \approx |\partial_- \hat{\vee}_p^{D,h}(p)|$$

where $\hat{\vee}_p^{D,h}(p)$ is the standard cone in R^n (vertex p, base D and height h).

- Consider the Dirichlet problem

$$\det[D_x^2 c(x, T_u(x)) - D^2 u(x)] = |D_{xy}^2 c| \cdot \frac{f(x)}{g(T_u(x))} \quad \text{in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

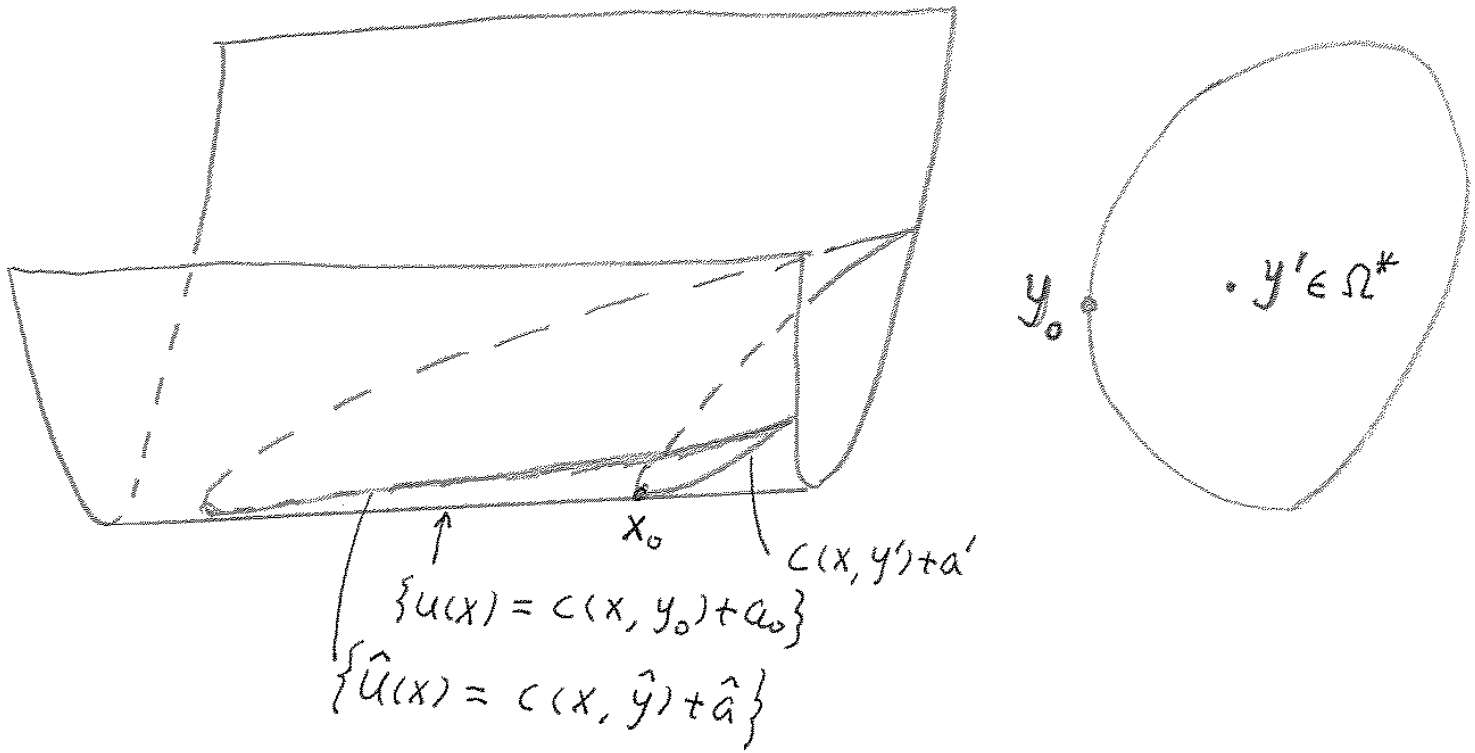
By the estimate of $|\partial_- \vee(p)|$, we have (if $\text{diam}(\Omega) < \delta_0$)

$$|u(x_c)| \approx |\Omega|^{2/n} \qquad x_c \text{ is the centre of } \Omega,$$

$$|u(x_0)| \approx d_{x_0}^* |\Omega|^{2/n} \qquad x_0 \text{ is any point in } \Omega,$$

where $d_{x_0}^*$ is a power of the distance from x_0 to $\partial\Omega$ after normalization.

- If the potential function is not strictly c-convex, there exists $x_0 \in \Omega$ such that $u(x_0) \approx u(x_c)$ but d_{x_0} as small as we want.



We consider the function $\hat{u} = \max\{u, c(x, y') + a'\}$.

Caffarelli extended u to $\mathbb{R}^n$ and consider the level set of u.

Proof of the $C^{1,\alpha}$ estimate.

By the strict c-convexity it follows that $u \in C^1$.

Hence $\forall\ x_0 \in \Omega$, by normalization we assume that

$$u(x_0) = 0 \quad \text{and} \quad u \geq 0$$

Then the strict c-convexity implies

$$u(x) \geq C|x - x_0|^m \quad \text{near } x_0$$

for some $m \geq 2$. By duality it implies that dual potential function $v \in C^{1,\alpha}$ with $\alpha = \frac{1}{m-1}$.

A Brief summary

Through works of many people, we have the regularity and geometry of the optimal transportation:

Geometry of OT under A3w:

- geometry of the cost function under A3w (Loeper),
- normalization and convexity of level sets,
- c-cone close to convex cone,
- Strict c-convexity.

Regularity of OT under A3w:

- $C^{1,\alpha}$,
- $C^{2,\alpha}$,
- $W^{2,p}$.

Monge's Problem

Monge's cost function $c(x, y) = |x - y|$
does not satisfy condition A1 (for existence of optimal maps)

A1: $\forall\, x, \xi \in R^n,\ \exists_1\, y \in R^n$ s.t. $\xi = D_x c(x, y)$.

(and $\forall\, y, \xi \in R^n,\ \exists_1\, x \in R^n$ s.t. $\xi = D_y c(x, y)$.)

nor it satisfies assumptions A2 and A3 (for the regularity)

A2: $|D^2_{xy} c| \neq 0$.

A3: $\exists\, c_0 > 0$ s.t. $\forall\, \xi, \eta \in R^n,\ \xi \perp \eta$

$$\sum (c_{ij,rs} - c^{p,q} c_{ij,p} c_{q,rs}) c^{r,k} c^{s,l} \xi_i \xi_j \eta_k \eta_l \geq c_0 |\xi|^2 |\eta|^2,$$

It is at the borderline of these conditions:

$$|x - y| = \lim_{\varepsilon \to 0} \sqrt{\varepsilon^2 + |x - y|^2} = \lim_{\varepsilon \to 0} |x - y|^{1-\varepsilon}.$$

Existence of optimal mappings in Monge's Problem

The existence was extensively studied in history.

- Evans-Gangbo – Mem AMS, p-Laplace equation $p \to \infty$
  (under some regularity conditions)
- Caffarelli-Feldman-McCann (JAMS2002),
  Trudinger-Wang (Calc Var 2001).
- Sudakov-Ambrosio – probability approach

Uniqueness in Monge's problem

The optimal mapping in Monge's problem is not unique. In fact, for Monge's problem in R^1 from $\Omega = [0, 1]$ to $\Omega^* = [1, 2]$, all mappings have the same total cost.

But by McCann-Feldman, there is a unique optimal mapping which is monotone, namely

$$(y - x) \cdot (s(y) - s(x)) \geq 0$$

Regularity in Monge's problem

We wish to know whether the monotone mapping is smooth. This is a rather difficult problem. Recall that Monge's cost

$$c(x,y) = |x-y| = \lim_{\varepsilon\to 0}[\varepsilon^2 + |x-y|^2]^{1/2} =: c_\varepsilon(x,y)$$

We wish to establish uniform estimates for potential functions u_ε and optimal mappings w.r.t. the cost function c_ε, as $\varepsilon \searrow 0$.

The potential function u_ε satisfies the equation

$$\det\Big[\frac{\sqrt{1-|Du|^2}}{\varepsilon}\big(\delta_{ij} - u_{x_i}u_{x_j}\big) - u_{x_ix_j}\Big] = \frac{[1-|Du|^2]^{\frac{n+2}{2}}}{\varepsilon^n}\,\frac{f}{g}.$$

When $\varepsilon > 0$ is small, it is a strongly singular equation.

Denote

$$w_{ij} = \frac{\sqrt{1-|Du|^2}}{\varepsilon}[\delta_{ij} - u_{x_i}u_{x_j}] - D^2 u_{x_i x_j}$$

$$A_{ij} = \frac{\sqrt{1-|Du|^2}}{\varepsilon}[\delta_{ij} - u_{x_i}u_{x_j}]$$

Then (note that $T = T_\varepsilon$, $W = W_\varepsilon$)

$$DT = [D^2_{xy}c]^{-1} W, \quad W = (w_{ij})$$

Denote $(A^{\alpha\beta})$ is the inverse of (A_{ij}) and

$$G = \sum A^{\alpha\beta} w_{\alpha\beta}$$

Denote $\lambda_i \geq 0$ the eigenvalues of DT.

The following result was obtained by Qi-Rui Li, Filippo Santambrogio and myself.

Theorem: If $u \in C^4$ is a solution, then

$$\operatorname{dist}(x, \partial\Omega)\, \lambda_i \leq C, \qquad (**)$$

where C is independent of $\varepsilon > 0$.

Proof: Consider the auxiliary function ηG, where η is a cut-off function such that ηG attains its maximum at some interior point. By very long computation, we obtain $\eta G \leq C$.

The above theorem implies

- Eigenvalues of DT is bounded.
- D^2u is bounded in the region $\{x \in \Omega : \ |T(x) - x| > 0\}$.
- $\partial_\nu T^\nu$ and $\partial_\xi T^\xi$ are bounded, where $\nu = \frac{Du}{|Du|}$, ξ is unit vector and $\xi \perp \nu$.

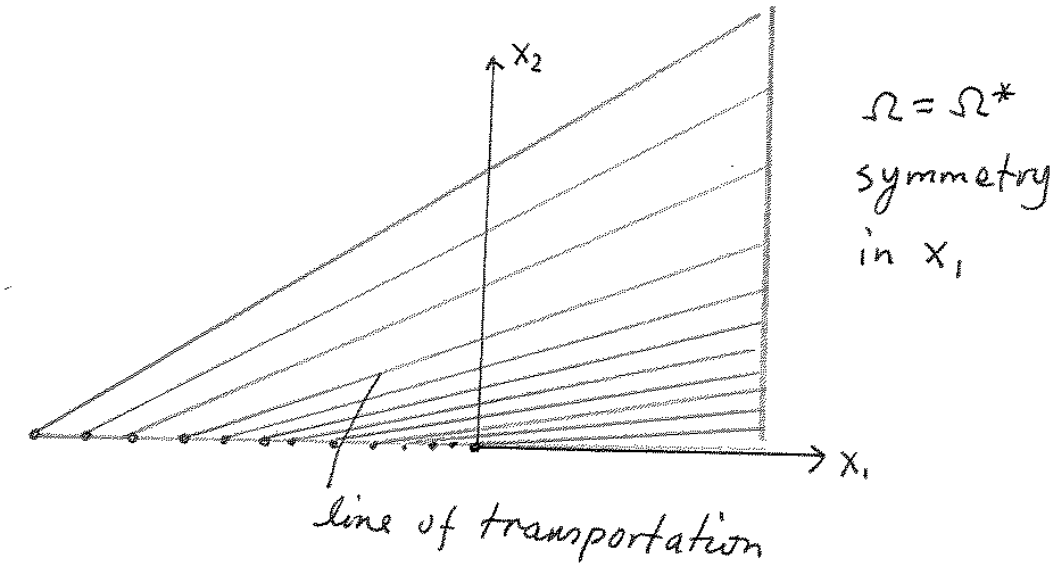
Question: Do we have the uniform estimate

$$|DT_\varepsilon| < C? \qquad (*)$$

i.e.

$$|\partial_\xi T^\nu| \leq C?$$

Answer: No. There exist convex smooth domains and smooth, positive mass distributions f, g such that $|DT_\varepsilon|$ is not uniformly bounded.



$\Omega = \Omega^*$ symmetry in x_1

$\exists$ density f, g such that in any subset of transport lines,

$$\int f = \int g.$$

and $\exists\ u \in C^\infty(\Omega \cap \{x_2 > 0\})$ s.t

$|Du| \leq 1$ and

$|u(p) - u(q)| = |p - q|$ if P, Q on the same line.

Question: Is T_ε uniformly continuous in ε if the target domain Ω^* is star-shaped with respect to any point of Ω.

We conjecture the answer is affirmative.

This problem is similar to the C^1 regularity of the ∞-Laplace equation, which is another open problem attracted much attention in the last two decades.

Fragala-Gelli-Pratelli studied the case in dim 2.
Assume that Ω and Ω^* are convex, $\overline{\Omega} \cap \overline{\Omega}^* = \emptyset$. Then the (monotone) optimal mapping is continuous .

Thank you