Gradient Flow in the 2-Wasserstein Metric

a Crandall and Liggett type proof of the exponential formula

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Gradient flow and discrete gradient flow

- Consider $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ and $E : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ proper, lower semicontinuous, coercive, and convex

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$$W_2(\mu, \nu) = \left(\int |\mathbf{id} - \mathbf{t}_\mu^\nu|^2 d\mu \right)^{1/2}$$

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$\implies x_n$ is a critical point of $y \mapsto \frac{1}{2\tau}|y - x_{n-1}|^2 + E(y)$

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- Analogously, define the discrete gradient flow sequence of E in W_2 as

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- To compensate for this, I follow [AGS] and require E to be convex along generalized geodesics (a class of curves in $\mathcal{P}_2(\mathbb{R}^d)$—will define soon).

Gradient flow and discrete gradient flow

- With this assumption, [AGS, Theorem 4.1.2] showed

$$E(\mu) < \infty \implies \forall \tau > 0 \,, \exists! \text{ minimizer of } \nu \mapsto \frac{1}{2\tau} W_2^2(\mu, \nu) + E(\nu)$$

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- Write the minimizer as $J_\tau \mu$ and call $J_\tau : \mu \mapsto J_\tau \mu$ the proximal map with time step τ.
  The nth term of the discrete gradient flow sequence with time step τ is $J_\tau^n \mu$.

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Key elements of Crandall and Liggett's method

Banach space $(X, ||\cdot||)$

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3. Combine these to get a recursive inequality:

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||J_\tau^n x - J_h^m x|| &= \left|\left| J_h \left[\frac{\tau - h}{\tau} J_\tau^n x + \frac{h}{\tau} J_\tau^{n-1} x\right] - J_h^m x \right|\right| && \text{by } ② \\
&\leq \left|\left| \left[\frac{\tau - h}{\tau} J_\tau^n x + \frac{h}{\tau} J_\tau^{n-1} x\right] - J_h^{m-1} x \right|\right| && \text{by } ① \\
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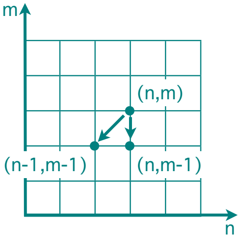

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1) Contraction Inequality

The W_2 analogue of $||J_\tau x - J_\tau y|| \leq ||x - y||$ is

Almost contraction inequality [C.]

$W_2^2(J_\tau\mu, J_\tau\nu) \leq W_2^2(\mu,\nu) + \tau^2|\partial E|^2(\mu)$

where $|\partial E|(u) := \limsup_{v\to u} \frac{(E(u)-E(v))^+}{d(u,v)}$ is the metric slope.

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This is proved using the discrete variational inequality [AGS, Theorem 4.1.2]:

$$\frac{1}{2\tau}[W_2^2(J_\tau \mu, \nu) - W_2^2(\mu, \nu)] \leq E(\nu) - E(J_\tau \mu) - \frac{1}{2\tau} W_2^2(\mu, J_\tau \mu)$$

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Proximal maps at different time steps [Jost, Mayer, C.]

If $0 < h \leq \tau$, $J_\tau \mu = J_h \left[\left(\frac{\tau-h}{\tau} \mathbf{t}_\mu^{J_\tau \mu} + \frac{h}{\tau} \mathbf{id}\right) \# \mu\right] = J_h \left[\mu^{\mu \to J_\tau \mu}_{\frac{\tau-h}{\tau}}\right]$

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This is analogous to the Banach space result since, for $\alpha \in [0,1]$,

$\alpha J_\tau x + (1-\alpha)x$ is the Banach space geodesic from x to $J_\tau x$ at time α

$(\alpha\mathbf{t}_\mu^{J_\tau\mu} + (1-\alpha)\mathbf{id})\#\mu$ is the Wasserstein geodesic from μ to $J_\tau\mu$ at time α.

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This follows from the Euler-Lagrange equation characterizing the minimizer $J_\tau\mu$.

Euler-Lagrange equation [AGS, C.]

ν is the unique minimizer of $\frac{1}{2\tau}W_2^2(\mu,\nu) + E(\nu)$ if and only if $\frac{1}{\tau}(\mathbf{t}_\nu^\mu - \mathbf{id}) \in \partial E(\nu)$ is a strong subdifferential.

Euler-Lagrange equation

Definition (Subdifferential)

$\boldsymbol{\xi} \in L^2(\mu)$ belongs to $\partial E(\mu)$ in case

$$E(\nu) - E(\mu) \geq \int \langle \boldsymbol{\xi}, \mathbf{t}_\mu^\nu - \mathbf{id} \rangle d\mu + o(W_2(\mu, \nu)) \quad \text{as } \nu \xrightarrow{W_2} \mu.$$

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If, in addition,

$$E(\mathbf{t}\#\mu) - E(\mu) \geq \int \langle \boldsymbol{\xi}, \mathbf{t} - \mathbf{id} \rangle d\mu + o(||\mathbf{t} - \mathbf{id}||_{L^2(\mu)}) \quad \text{as } \mathbf{t} \xrightarrow{L^2(\mu)} \mathbf{id}$$

$\boldsymbol{\xi}$ is a strong subdifferential.

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$$E(\mathbf{t}\#\mu) - E(\mu) \geq \int \langle \boldsymbol{\xi}, \mathbf{t} - \mathbf{id} \rangle d\mu + o(||\mathbf{t} - \mathbf{id}||_{L^2(\mu)}) \quad \text{as } \mathbf{t} \xrightarrow{L^2(\mu)} \mathbf{id}$$

$\boldsymbol{\xi}$ is a strong subdifferential.

[AGS, Lemma 10.1.2], following a method introduced by [Otto], prove

ν minimizes $\frac{1}{2\tau} W_2^2(\mu,\nu) + E(\nu) \implies \frac{1}{\tau}(\mathbf{t}_\nu^\mu - \mathbf{id}) \in \partial E(\nu)$ is a strong subdifferential.

Euler-Lagrange equation

To prove the converse,

$$\frac{1}{\tau}(\mathbf{t}_\nu^\mu - \mathbf{id}) \in \partial E(\nu) \text{ is a strong subdifferential } \implies \nu \text{ minimizes } \frac{1}{2\tau} W_2^2(\mu,\nu) + E(\nu),$$

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We'd like to use that $\frac{1}{\tau}(\mathbf{t}_\nu^\mu - \mathbf{id}) \in \partial E(\nu)$ to conclude $0 \in \partial \Phi_{\tau,\mu}(\nu)$, hence ν minimizes $\Phi_{\tau,\mu}$. However, this argument doesn't work since $\Phi_{\tau,\mu}$ is not convex.

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Again, we confront the problem that $\nu \mapsto W_2^2(\mu,\nu)$ is not convex.

Goal

1. Contraction inequality: $W_2^2(J_\tau\mu, J_\tau\nu) \leq W_2^2(\mu,\nu) + \tau^2|\partial E|^2(\mu)$
2. Proximal maps at different time steps: if $0 < h \leq \tau$, $J_\tau x = J_h\left[\frac{\tau-h}{\tau}J_\tau x + \frac{h}{\tau}x\right]$
   Will come back to this.
3. Combine these to get a recursive inequality:

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3) Recursive Inequality

Combining

1. the almost contraction inequality and
2. the relation between proximal maps at different time steps

we get

$$\begin{aligned} W_2^2(J_\tau^n\mu, J_h^m\mu) &= W_2^2(J_h(\mu_{\frac{\tau-h}{\tau}}^{J_\tau^{n-1}\mu\to J_\tau^n\mu}), J_h^m\mu) && \text{by ②} \\ &\leq W_2^2(\mu_{\frac{\tau-h}{\tau}}^{J_\tau^{n-1}\mu\to J_\tau^n\mu}, J_h^{m-1}\mu) + h^2|\partial E|^2(J_h^{m-1}\mu) && \text{by ①} \end{aligned}$$

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To follow the Banach space case, we would like to bound the first term by a convex combination of $W_2^2(J_\tau^{n-1}\mu, J_h^{m-1}\mu)$ and $W_2^2(J_\tau^n\mu, J_h^{m-1}\mu)$, but this again requires the convexity of

$$\nu \mapsto W_2^2(\mu, \nu) \ .$$

Generalized geodesics

[AGS] ran into the same problem. Their solution was to consider convexity along a different class of curves. While $\nu \mapsto W_2^2(\mu, \nu)$ is not convex along all geodesics, it is convex along all generalized geodesics with base μ [AGS, Lemma 9.2.1].

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Definition (Generalized geodesic)

The generalized geodesic from μ_0 to μ_1 with base ω is

$$\mu_\alpha := ((1-\alpha)\mathbf{t}_\omega^{\mu_0} + \alpha \mathbf{t}_\omega^{\mu_1})\,\#\omega.$$

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On its own, the convexity of $\nu \mapsto W_2^2(\mu, \nu)$ along generalized geodesics with base μ is not enough to control

$$W_2^2(\mu_{\frac{\tau-h}{\tau}}^{J_\tau^{n-1}\mu \to J_\tau^n \mu}, J_h^{m-1}\mu) ,$$

since $\mu_{\frac{\tau-h}{\tau}}^{J_\tau^{n-1}\mu \to J_\tau^n \mu}$ is not a generalized geodesic with base $J_h^{m-1}\mu$. For this reason, consider a related notion, also introduced by [AGS, Equation 7.3.2].

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Given ω, the transport metric from μ to ν with base ω is

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We want to prove

$$\frac{1}{\tau}(\mathbf{t}_\nu^\mu - \mathbf{id}) \in \partial E(\nu) \text{ is a strong subdifferential } \implies \nu \text{ minimizes } \frac{1}{2\tau} W_2^2(\mu,\nu) + E(\nu),$$

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To do this, define what it means for $\boldsymbol{\xi} \in \partial_{2,\omega} E(\mu)$ and show

Lemma ([C.])

1. *E convex in $W_{2,\omega}$ and $0 \in \partial_{2,\omega} E(\mu) \implies \mu$ minimizes E.*
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- By 2, $\frac{1}{\tau}(\mathbf{id} - \mathbf{t}_\mu^\nu) \in \partial_{2,\mu} E(\nu)$

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- $\Phi_{\tau,\mu}$ is convex in $W_{2,\mu} \implies \nu$ must be a minimizer, by 1

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Proximal maps at different time steps

Thus, we have shown,

Euler-Lagrange equation [AGS, C.]

ν is the unique minimizer of $\frac{1}{2\tau}W_2^2(\mu,\nu) + E(\nu)$ if and only if $\frac{1}{\tau}(\mathbf{t}_\nu^\mu - \mathbf{id}) \in \partial E(\nu)$ is a strong subdifferential.

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If $0<h\le\tau$, $J_\tau\mu=J_h\left[\left(\frac{\tau-h}{\tau}\mathbf{t}_\mu^{J_\tau\mu}+\frac{h}{\tau}\mathbf{id}\right)\#\mu\right]=J_h\left[\mu^{\mu\to J_\tau\mu}_{\frac{\tau-h}{\tau}}\right]$

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||J_\tau^n x - J_h^m x|| &= \left|\left|J_h\left[\frac{\tau-h}{\tau}J_\tau^n x + \frac{h}{\tau}J_\tau^{n-1}x\right] - J_h^m x\right|\right| && \text{by } ❷ \\
&\leq \left|\left|\frac{\tau-h}{\tau}J_\tau^n x + \frac{h}{\tau}J_\tau^{n-1}x - J_h^{m-1}x\right|\right| && \text{by } ❶ \\
&\leq \frac{\tau-h}{\tau}||J_\tau^n x - J_h^{m-1}x|| + \frac{h}{\tau}||J_\tau^{n-1}x - J_h^{m-1}x||
\end{aligned}$$

Recursive Inequality

Proof.

Combining the contraction inequality with the relation between proximal maps at different time steps, we get

$$
\begin{aligned}
&W_2^2(J_\tau^n\mu, J_h^m\mu)\\
&= W_2^2(J_h(\mu_{\frac{\tau-h}{\tau}}^{J_\tau^{n-1}\mu\to J_\tau^n\mu}), J_h^m\mu)\\
&\le W_2^2(\mu_{\frac{\tau-h}{\tau}}^{J_\tau^{n-1}\mu\to J_\tau^n\mu}, J_h^{m-1}\mu) + h^2|\partial E|^2(J_h^{m-1}\mu)\\
&\le W_{2,J_\tau^{n-1}\mu}^2(\mu_{\frac{\tau-h}{\tau}}^{J_\tau^{n-1}\mu\to J_\tau^n\mu}, J_h^{m-1}\mu) + h^2|\partial E|^2(J_h^{m-1}\mu)\\
&\le \frac{h}{\tau}W_{2,J_\tau^{n-1}\mu}^2(J_\tau^{n-1}\mu, J_h^{m-1}\mu) + \frac{\tau-h}{\tau}W_{2,J_\tau^{n-1}\mu}^2(J_\tau^n\mu, J_h^{m-1}\mu) + h^2|\partial E|^2(J_h^{m-1}\mu)\\
&= \frac{h}{\tau}W_2^2(J_\tau^{n-1}\mu, J_h^{m-1}\mu) + \frac{\tau-h}{\tau}W_{2,J_\tau^{n-1}\mu}^2(J_\tau^n\mu, J_h^{m-1}\mu) + h^2|\partial E|^2(J_h^{m-1}\mu)
\end{aligned}
$$

Now we need to go from the transport metric back to the W_2 metric.

Recursive Inequality

Lemma (Controlling transport metric in terms of W_2 [C.])

$$W^2_{2,J_\tau^{n-1}\mu}(J_h^{m-1}\mu, J_\tau^n\mu) \leq \frac{\tau}{h}\left(W_2^2(J_h^{m-1}\mu, J_\tau^n\mu) - W_2^2(J_h^m\mu, J_\tau^n\mu)\right) + W_2^2(J_\tau^{n-1}\mu, J_h^{m-1}\mu) + \tau h|\partial E|^2(J_h^{m-1}\mu)$$

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We can now finish the proof.

Proof.

Plugging this into the previous inequality, rearranging, and simplifying gives

$$W_2^2(J_\tau^n\mu, J_h^m\mu) \leq \frac{h}{\tau}W_2^2(J_h^{m-1}\mu, J_\tau^{n-1}\mu) + \frac{\tau - h}{\tau}W_2^2(J_h^{m-1}\mu, J_\tau^n\mu) + 2h^2|\partial E|^2(\mu)$$

□

Goal

1. Contraction inequality: $W_2^2(J_\tau\mu, J_\tau\nu) \leq W_2^2(\mu,\nu) + \tau^2|\partial E|^2(\mu)$
2. Proximal maps at different time steps: if $0 < h \leq \tau$,
   $J_\tau\mu = J_h\left[\left(\frac{\tau-h}{\tau}\mathbf{t}_\mu^{J_\tau\mu} + \frac{h}{\tau}\mathbf{id}\right)\#\mu\right] = J_h\left[\mu^{\mu\to J_\tau\mu}_{\frac{\tau-h}{\tau}}\right]$
3. Combine these to get a recursive inequality:

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||J_\tau^n x - J_h^m x|| &= \left|\left|J_h\left[\frac{\tau-h}{\tau}J_\tau^n x + \frac{h}{\tau}J_\tau^{n-1}x\right] - J_h^m x\right|\right| && \text{by } 2 \\
&\leq \left|\left|\frac{\tau-h}{\tau}J_\tau^n x + \frac{h}{\tau}J_\tau^{n-1}x - J_h^{m-1}x\right|\right| && \text{by } 1 \\
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- Iterating the recursive inequality in a manner similar to [Rasmussen] gives

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- Therefore, taking $\tau = \frac{t}{n}$ and $h = \frac{t}{m}$ with $m \geq n$, so $h \leq \tau$, we obtain

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- This rate improves upon the metric space result of [Clément, Desch]
  $W_2(J_{t/n}^n \mu, \mu(s)) \leq O\left(\frac{1}{n^{1/4}}\right) |\partial E|(\mu)$
  (though their result holds in greater generality)

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