Spaces with Ricci curvature bounded from below

Nicola Gigli

August 29, 2013

Lessons

1) On the definition of spaces with Ricci curvature bounded from below

2) Analytic properties of $RCD(K, N)$ spaces

3) Geometric properties of $RCD(K, N)$ spaces

Quoting the first sentence of Cheng-Yau ’75

‘Most of the problems in differential geometry can be reduced to problems in differential equations on Riemannian manifolds’

Few things to forget about

- Forget about Lipschitz functions
- Forget about charts
- Forget (for a second) about defining who tangent/cotangent vectors are: focus on defining $\nabla f \cdot \nabla g$ for Sobolev f, g

Analytic properties of $RCD(K, N)$ spaces

- Differential calculus on infinitesimally Hilbertian spaces
- The heat flow on $RCD(K, \infty)$ spaces again
- Bochner inequality
- Optimal maps
- Distributional Laplacian

Analytic properties of $RCD(K, N)$ spaces

- Differential calculus on infinitesimally Hilbertian spaces
- The heat flow on $RCD(K, \infty)$ spaces again
- Bochner inequality
- Optimal maps
- Distributional Laplacian

Infinitesimally Hilbertian spaces and the object $\nabla f \cdot \nabla g$

We say that $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian if $W^{1,2}(X)$ is Hilbert.

Infinitesimally Hilbertian spaces and the object $\nabla f \cdot \nabla g$

We say that $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian if $W^{1,2}(X)$ is Hilbert.

Let $(X, \mathsf{d}, \mathfrak{m})$ be inf. Hilb. and $f, g \in S^2(X)$.

Infinitesimally Hilbertian spaces and the object $\nabla f \cdot \nabla g$

We say that $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian if $W^{1,2}(X)$ is Hilbert.

Let $(X, \mathsf{d}, \mathfrak{m})$ be inf. Hilb. and $f, g \in S^2(X)$.

We define $\nabla f \cdot \nabla g : X \to \mathbb{R}$ as

$$\nabla f \cdot \nabla g := \inf_{\varepsilon > 0} \frac{|D(g + \varepsilon f)|^2 - |Dg|^2}{2\varepsilon}$$

Calculus rules

Thm. (G. '12. Ambrosio, G., Savaré '11) For $(X, \mathsf{d}, \mathfrak{m})$ inf. Hilb. and $f, g \in S^2(X)$ we have

Cauchy-Schwarz $\quad |\nabla f \cdot \nabla g| \leq |Df||Dg| \in L^1(X, \mathfrak{m})$

Locality $\quad \nabla f \cdot \nabla g = \nabla \tilde{f} \cdot \nabla \tilde{g} \quad \mathfrak{m}$-a.e. on $\{f = \tilde{f}\} \cap \{g = \tilde{g}\}$.

Linearity $\quad \nabla(\alpha_0 f_0 + \alpha_1 f_1) \cdot \nabla g = \alpha_0 \nabla f_0 \cdot \nabla g + \alpha_1 \nabla f_1 \cdot \nabla g$

Chain rule $\quad \nabla(\varphi \circ f) = \varphi' \circ f \, \nabla f \cdot \nabla g \quad$ for φ Lipschitz

Leibniz rule $\quad \nabla(f_1 f_2) \cdot \nabla g = f_1 \nabla f_2 \cdot \nabla g + f_2 \nabla f_1 \cdot \nabla g$.

Symmetry $\quad \nabla f \cdot \nabla g = \nabla g \cdot \nabla f$

Plan representing gradients: definition

For $g \in S^2$ and $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ test plan it holds

$$\varlimsup_{t \downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} \,\mathrm{d}\boldsymbol{\pi} \leq \frac{1}{2} \int |Dg|^2(\gamma_0) \,\mathrm{d}\boldsymbol{\pi} + \varlimsup_{t \downarrow 0} \frac{1}{2t} \iint_0^t |\dot{\gamma}_s|^2 \,\mathrm{d}s \,\mathrm{d}\boldsymbol{\pi}$$

Plan representing gradients: definition

For $g \in S^2$ and $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ test plan it holds

$$\varlimsup_{t\downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} \,\mathrm{d}\boldsymbol{\pi} \leq \frac{1}{2} \int |Dg|^2(\gamma_0)\,\mathrm{d}\boldsymbol{\pi} + \varlimsup_{t\downarrow 0} \frac{1}{2t} \iint_0^t |\dot{\gamma}_s|^2 \,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}$$

We say that $\boldsymbol{\pi}$ *represents* ∇g, provided it holds

$$\varliminf_{t\downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} \,\mathrm{d}\boldsymbol{\pi} \geq \frac{1}{2} \int |Dg|^2(\gamma_0)\,\mathrm{d}\boldsymbol{\pi} + \varlimsup_{t\downarrow 0} \frac{1}{2t} \iint_0^t |\dot{\gamma}_s|^2 \,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}$$

Plan representing gradients: existence

Thm (G. '12. Ambrosio, G., Savaré '11. G., Kuwada, Ohta '10).
For $g \in S^2(X)$ and $\mu \in \mathscr{P}(X)$ such that $\mu \leq C\mathfrak{m}$, a plan $\boldsymbol{\pi}$ representing ∇g and such that ${\mathrm{e}_0}_\sharp\boldsymbol{\pi} = \mu$ exists.

First order differentiation formula

Let $f, g \in S^2(X)$, and $\boldsymbol{\pi}$ which represents ∇g.

First order differentiation formula

Let $f, g \in S^2(X)$, and $\boldsymbol{\pi}$ which represents ∇g.

Then

$$\lim_{t \downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi} = \int \nabla f \cdot \nabla g \, (\gamma_0) \, \mathrm{d}\boldsymbol{\pi}$$

Analytic properties of $RCD(K, N)$ spaces

- Differential calculus on infinitesimally Hilbertian spaces
- The heat flow on $RCD(K, \infty)$ spaces again
- Bochner inequality
- Optimal maps
- Distributional Laplacian

A property of GF of K-convex functions on $\mathbb{R}^d$

Let $E : \mathbb{R}^d \to \mathbb{R}$ be K-convex and $t \mapsto x_t$ be such that

$$x_t' = -\nabla E(x_t).$$

A property of GF of K-convex functions on $\mathbb{R}^d$

Let $E : \mathbb{R}^d \to \mathbb{R}$ be K-convex and $t \mapsto x_t$ be such that

$$x_t' = -\nabla E(x_t).$$

Pick $y \in \mathbb{R}^d$ and notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|x_t - y|^2 = x_t' \cdot (x_t - y) = \nabla E(x_t) \cdot (y - x_t)$$

and for $y_{t,s} := (1-s)x_t + sy$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(y_{t,s}) = \nabla E(x_t) \cdot (y - x_t).$$

A property of GF of K-convex functions on $\mathbb{R}^d$

Let $E : \mathbb{R}^d \to \mathbb{R}$ be K-convex and $t \mapsto x_t$ be such that

$$x'_t = -\nabla E(x_t).$$

Pick $y \in \mathbb{R}^d$ and notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|x_t - y|^2 = x'_t \cdot (x_t - y) = \nabla E(x_t) \cdot (y - x_t)$$

and for $y_{t,s} := (1-s)x_t + sy$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0} E(y_{t,s}) = \nabla E(x_t) \cdot (y - x_t).$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|x_t - y|^2 \leq E(y) - E(x_t) - \frac{K}{2}|x_t - y|^2$$

EVI_K gradient flows

Def. On a metric space (Y, d_Y), we say that $(x_t) \subset Y$ is an EVI_K-GF of $E : Y \to [0, \infty]$ if it is loc. abs. cont. and for every $y \in Y$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathrm{d}^2(x_t, y) \leq E(y) - E(x_t) - \frac{K}{2}\mathrm{d}^2(x_t, y), \qquad \textit{a.e. } t > 0$$

EVI_K gradient flows

Def. On a metric space (Y, d_Y), we say that $(x_t) \subset Y$ is an EVI_K-GF of $E : Y \to [0, \infty]$ if it is loc. abs. cont. and for every $y \in Y$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathrm{d}^2(x_t, y) \leq E(y) - E(x_t) - \frac{K}{2}\mathrm{d}^2(x_t, y), \qquad a.e.\ t > 0$$

(Savaré) If (x_t) is an EVI_K gradient flows it satisfies

$$E(x_0) = E(x_t) + \frac{1}{2}\int_0^t |x_s'|^2 + |\partial^- E|^2(x_s)\,\mathrm{d}s, \qquad \forall t > 0$$

The viceversa is not true

The heat flow as EVI_K gradient flow of the entropy

We want to prove that the heat flow is an EVI_K gradient flow of the entropy.

The heat flow as EVI_K gradient flow of the entropy

We want to prove that the heat flow is an EVI_K gradient flow of the entropy.

Thus let $t \mapsto \mu_t = \rho_t \mathfrak{m}$ be an heat flow and $\nu = \eta \mathfrak{m}$ given.

The heat flow as EVI_K gradient flow of the entropy

We want to prove that the heat flow is an EVI_K gradient flow of the entropy.

Thus let $t \mapsto \mu_t = \rho_t \mathfrak{m}$ be an heat flow and $\nu = \eta \mathfrak{m}$ given.

We want to compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\mathrm{Ent}_{\mathfrak{m}}(\nu_{t,s})$$

where $s \mapsto \nu_{t,s}$ is a geodesic joining μ_t to ν.

Derivative of $\frac{1}{2}W_2^2(\mu_t,\nu)$

Fix t_0 a point of differentiability of $t \mapsto \frac{1}{2}W_2^2(\mu_t,\nu)$ and let φ be a Kantorovich potential from μ_{t_0} to ν.
Then

$$\frac{1}{2}W_2^2(\mu_{t_0},\nu) = \int \varphi \,\mathrm{d}\mu_{t_0} + \int \varphi^c \,\mathrm{d}\nu$$

$$\frac{1}{2}W_2^2(\mu_{t_0+h},\nu) \geq \int \varphi \,\mathrm{d}\mu_{t_0+h} + \int \varphi^c \,\mathrm{d}\nu$$

Recalling that $\mu_t = \rho_t \mathfrak{m}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} \frac{1}{2}W_2^2(\mu_t,\nu) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} \int \varphi \,\mathrm{d}\mu_t = \int \varphi \Delta\rho_{t_0} \,\mathrm{d}\mathfrak{m}$$

Some properties of W_2-geodesics

Thm. (Regularity of interpolated densities Rajala '12)
Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $CD(K, \infty)$ space and $\mu, \nu \in \mathscr{P}(X)$ s.t. $\mu, \nu \leq C\mathfrak{m}$.

Some properties of W_2-geodesics

Thm. (Regularity of interpolated densities Rajala '12)
Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $CD(K, \infty)$ space and $\mu, \nu \in \mathscr{P}(X)$ s.t. $\mu, \nu \leq C\mathfrak{m}$.

Then there exists a geodesic (μ_t) such that $\mu_t \leq C'\mathfrak{m}$ for every $t \in [0, 1]$ and $t \mapsto \mathrm{Ent}_{\mathfrak{m}}(\mu_t)$ is K-convex.

Some properties of W_2-geodesics

Thm. (Regularity of interpolated densities Rajala '12)
Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $CD(K, \infty)$ space and $\mu, \nu \in \mathscr{P}(X)$ s.t. $\mu, \nu \leq C\mathfrak{m}$.

Then there exists a geodesic (μ_t) such that $\mu_t \leq C'\mathfrak{m}$ for every $t \in [0, 1]$ and $t \mapsto \mathrm{Ent}_{\mathfrak{m}}(\mu_t)$ is K-convex.

Thm. (Metric Brenier's theorem Ambrosio, G., Savaré '11) Let (μ_t) be a geodesic such that $\mu_t \leq C\mathfrak{m}$ for every $t \in [0, 1]$, $\boldsymbol{\pi} \in \mathscr{P}(C([0, 1], X))$ a lifting of it and φ a Kantorovich potential inducing it.

Some properties of W_2-geodesics

Thm. (Regularity of interpolated densities Rajala '12)
Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $CD(K, \infty)$ space and $\mu, \nu \in \mathscr{P}(X)$ s.t. $\mu, \nu \leq C\mathfrak{m}$.

Then there exists a geodesic (μ_t) such that $\mu_t \leq C'\mathfrak{m}$ for every $t \in [0, 1]$ and $t \mapsto \mathrm{Ent}_{\mathfrak{m}}(\mu_t)$ is K-convex.

Thm. (Metric Brenier's theorem Ambrosio, G., Savaré '11) Let (μ_t) be a geodesic such that $\mu_t \leq C\mathfrak{m}$ for every $t \in [0, 1]$, $\boldsymbol{\pi} \in \mathscr{P}(C([0, 1], X))$ a lifting of it and φ a Kantorovich potential inducing it.

Then $\boldsymbol{\pi}$ represents the gradient of $-\varphi$.

Derivative of $\mathrm{Ent}_{\mathfrak{m}}(\nu_s)$

Let $s \mapsto \nu_s$ be a geodesic s.t. $\nu_s \leq C\mathfrak{m}$ for every s and such that $\nu_0 = \eta\mathfrak{m}$ with $\eta \geq c > 0$, $\eta \in W^{1,2}(X)$.
Let φ be a Kantorovich potential inducing it.

Derivative of $\mathrm{Ent}_{\mathfrak{m}}(\nu_s)$

Let $s \mapsto \nu_s$ be a geodesic s.t. $\nu_s \leq C\mathfrak{m}$ for every s and such that $\nu_0 = \eta\mathfrak{m}$ with $\eta \geq c > 0$, $\eta \in W^{1,2}(X)$.
Let φ be a Kantorovich potential inducing it.
Then

$$
\begin{aligned}
\varliminf_{s\downarrow 0} \frac{\mathrm{Ent}_{\mathfrak{m}}(\nu_s) - \mathrm{Ent}_{\mathfrak{m}}(\nu_0)}{s} &\geq \varliminf_{s\downarrow 0} \frac{1}{s}\int \log\eta \,\mathrm{d}(\nu_s - \nu_0) \\
&= \varliminf_{s\downarrow 0} \int \frac{\log\eta(\gamma_s) - \log\eta(\gamma_0)}{s} \,\mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= -\int \nabla(\log\eta)\cdot\nabla\varphi(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= -\int \nabla(\log\eta)\cdot\nabla\varphi\;\eta\,\mathrm{d}\mathfrak{m} \\
&= -\int \nabla\eta\cdot\nabla\varphi\,\mathrm{d}\mathfrak{m}
\end{aligned}
$$

Derivative of $\mathrm{Ent}_{\mathfrak{m}}(\nu_s)$

Let $s \mapsto \nu_s$ be a geodesic s.t. $\nu_s \leq C\mathfrak{m}$ for every s and such that $\nu_0 = \eta\mathfrak{m}$ with $\eta \geq c > 0$, $\eta \in W^{1,2}(X)$.
Let φ be a Kantorovich potential inducing it.
Then

$$\begin{aligned}
\varliminf_{s\downarrow 0} \frac{\mathrm{Ent}_{\mathfrak{m}}(\nu_s) - \mathrm{Ent}_{\mathfrak{m}}(\nu_0)}{s} &\geq \varliminf_{s\downarrow 0} \frac{1}{s} \int \log\eta \,\mathrm{d}(\nu_s - \nu_0) \\
&= \varliminf_{s\downarrow 0} \int \frac{\log\eta(\gamma_s) - \log\eta(\gamma_0)}{s} \,\mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= -\int \nabla(\log\eta) \cdot \nabla\varphi(\gamma_0) \,\mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= -\int \nabla(\log\eta) \cdot \nabla\varphi \;\eta \,\mathrm{d}\mathfrak{m} \\
&= -\int \nabla\eta \cdot \nabla\varphi \,\mathrm{d}\mathfrak{m}
\end{aligned}$$

The heat flow is an EVI$_K$ gradient flow of the entropy

We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu) \le \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\mathrm{Ent}_{\mathfrak{m}}(\nu_{t,s})$$
$$\le \mathrm{Ent}_{\mathfrak{m}}(\nu) - \mathrm{Ent}_{\mathfrak{m}}(\mu_t) - \frac{K}{2}W_2^2(\mu_t,\nu)$$

The heat flow is an EVI_K gradient flow of the entropy

We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu) \le \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\mathrm{Ent}_{\mathfrak{m}}(\nu_{t,s})$$
$$\le \mathrm{Ent}_{\mathfrak{m}}(\nu) - \mathrm{Ent}_{\mathfrak{m}}(\mu_t) - \frac{K}{2}W_2^2(\mu_t,\nu)$$

We deduce that for $(\mu_t),(\nu_t) \subset \mathscr{P}(X)$ heat flows we have

$$W_2^2(\mu_t,\nu_t) \le e^{-2Kt}W_2^2(\mu_0,\nu_0)$$

Heat Kernel and Bronian motion

We deduce that there exists the heat flow $t \mapsto \mu_t[x]$ starting from δ_x for any $x \in X$.

General constructions related to the theory of Dirichlet forms then grant existence and uniqueness of a Markov process $\mathbf{X}_t$ with transition probabilities $\mu_t[x]$, i.e.:

$$\mathbb{P}(\mathbf{X}_{t+s} \in A | \mathbf{X}_t = x) = \mu_t[x](A)$$

Analytic properties of $RCD(K, N)$ spaces

- Differential calculus on infinitesimally Hilbertian spaces
- The heat flow on $RCD(K, \infty)$ spaces again
- Bochner inequality
- Optimal maps
- Distributional Laplacian

A duality result

Thm. (Kuwada '09)
Let $\mathsf{H}_t : \mathscr{P}(X) \to \mathscr{P}(X)$ be the heat flow at level of measures and $\mathsf{h}_t : L^1 \to L^1$ the one for densities.

A duality result

Thm. (Kuwada '09)
Let $\mathsf{H}_t : \mathscr{P}(X) \to \mathscr{P}(X)$ be the heat flow at level of measures and $\mathsf{h}_t : L^1 \to L^1$ the one for densities.
Then TFAE:

$$W_2^2(\mathsf{H}_t(\mu), \mathsf{H}_t(\nu)) \leq e^{-2Kt} W_2^2(\mu, \nu), \qquad \forall t \geq 0,\ \mu, \nu \in \mathscr{P}(X)$$

$$\mathrm{lip}^2(\mathsf{h}_t(f)) \leq e^{-2Kt}\, \mathsf{h}_t(\mathrm{lip}^2(f)), \qquad \forall t \geq 0,\ f : X \to \mathbb{R} \text{ Lipschitz}$$

where

$$\mathrm{lip}(f)(x) := \overline{\lim_{y \to x}} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)}$$

Density in energy in $W^{1,2}$ of Lipschitz functions

Thm. (Ambrosio, G., Savaré '11) Let $(X, \mathsf{d}, \mathfrak{m})$ be a mms.

Density in energy in $W^{1,2}$ of Lipschitz functions

Thm. (Ambrosio, G., Savaré '11) Let $(X, \mathsf{d}, \mathfrak{m})$ be a mms. Then:

- for every $(f_n) \subset \mathrm{LIP}(X)$converging in L^2 to some f, we have

$$|Df| \leq G, \qquad \text{where } G \text{ is any } L^2\text{-weak limit of } (\mathrm{lip}(f_n))$$

- for every $f \in W^{1,2}(X)$ there exists $(f_n) \subset \mathrm{LIP}(X)$ L^2-converging to f such that

$$|Df| = \lim_n \mathrm{lip}(f_n) \qquad \text{the limit being intended strong in } L^2$$

Bochner inequality ($N = \infty$)

(G., Kuwada, Ohta '10. Ambrosio, G., Savaré '11)
Starting from

$$\mathrm{lip}^2(\mathsf{h}_t(f)) \leq e^{-2Kt}\,\mathsf{h}_t(\mathrm{lip}^2(f)), \qquad \forall t \geq 0,\ f \in \mathrm{LIP}(X)$$

Bochner inequality ($N = \infty$)

(G., Kuwada, Ohta '10. Ambrosio, G., Savaré '11)
Starting from

$$\mathrm{lip}^2(\mathsf{h}_t(f)) \leq e^{-2Kt}\,\mathsf{h}_t(\mathrm{lip}^2(f)), \qquad \forall t \geq 0,\ f \in \mathrm{LIP}(X)$$

and by relaxation we deduce

$$|D\mathsf{h}_t(f)|^2 \leq e^{-2Kt}\mathsf{h}_t(|Df|^2) \qquad \forall t \geq 0,\ f \in W^{1,2}(X)$$

Bochner inequality ($N = \infty$)

(G., Kuwada, Ohta '10. Ambrosio, G., Savaré '11)
Starting from

$$\mathrm{lip}^2(\mathsf{h}_t(f)) \le e^{-2Kt}\,\mathsf{h}_t(\mathrm{lip}^2(f)), \qquad \forall t \ge 0,\ f \in \mathrm{LIP}(X)$$

and by relaxation we deduce

$$|D\mathsf{h}_t(f)|^2 \le e^{-2Kt}\mathsf{h}_t(|Df|^2) \qquad \forall t \ge 0,\ f \in W^{1,2}(X)$$

which gives

$$\int \Delta g \frac{|Df|^2}{2}\,\mathrm{d}\mathfrak{m} \ge \int (\nabla f \cdot \nabla \Delta f + K|Df|^2) g\,\mathrm{d}\mathfrak{m}$$

for every $f \in W^{1,2}(X) \cap D(\Delta)$ with $\Delta f \in W^{1,2}(X)$ and $g \in L^\infty(X) \cap D(\Delta)$ with $g \ge 0$ and $\Delta g \in L^\infty(X)$.

Bochner inequality ($N = \infty$)

(G., Kuwada, Ohta '10. Ambrosio, G., Savaré '11)
Starting from

$$\mathrm{lip}^2(\mathsf{h}_t(f)) \le e^{-2Kt}\,\mathsf{h}_t(\mathrm{lip}^2(f)), \qquad \forall t \ge 0,\ f \in \mathrm{LIP}(X)$$

and by relaxation we deduce

$$|D\mathsf{h}_t(f)|^2 \le e^{-2Kt}\mathsf{h}_t(|Df|^2) \qquad \forall t \ge 0,\ f \in W^{1,2}(X)$$

which gives

$$\int \Delta g \frac{|Df|^2}{2}\,\mathrm{d}\mathfrak{m} \ge \int (\nabla f \cdot \nabla \Delta f + K|Df|^2) g\,\mathrm{d}\mathfrak{m}$$

for every $f \in W^{1,2}(X) \cap D(\Delta)$ with $\Delta f \in W^{1,2}(X)$ and $g \in L^\infty(X) \cap D(\Delta)$ with $g \ge 0$ and $\Delta g \in L^\infty(X)$.

Also the converse implication from Bochner to $RCD(K,\infty)$ holds (Ambrosio, G., Savaré '12)

Bochner inequality ($N < \infty$)

(Erbar, Kuwada, Sturm '13) On an $RCD(K, N)$ space we have

$$\int \Delta g \frac{|Df|^2}{2} \, \mathrm{d}\mathfrak{m} \geq \int \Big(\frac{(\Delta f)^2}{N} + \nabla f \cdot \nabla \Delta f + K|Df|^2 \Big) g \, \mathrm{d}\mathfrak{m}$$

(see also (Ambrosio, Mondino, Savaré - in progress))

Related results

(Mondino, Garofalo '13) Li-Yau inequality: for $f \geq 0$ on $RCD(0, N)$ spaces we have

$$\Delta(\log(\mathsf{h}_t f)) \geq \frac{N}{2t}$$

(Kell '13, Jiang '11, Koskela, Rajala, Shanmugalingam '03) Local Lipschitz regularity of harmonic functions on $RCD(K, N)$ spaces

Analytic properties of $RCD(K, N)$ spaces

- Differential calculus on infinitesimally Hilbertian spaces
- The heat flow on $RCD(K, \infty)$ spaces again
- Bochner inequality
- Optimal maps
- Distributional Laplacian

Optimal maps

Thm. (G., Rajala, Sturm '13) Let $(X, \mathsf{d}, \mathfrak{m})$ be $RCD(K, N)$, $\mu, \nu \in \mathscr{P}(X)$ with $\mu \ll \mathfrak{m}$.

Optimal maps

Thm. (G., Rajala, Sturm '13) Let $(X, \mathsf{d}, \mathfrak{m})$ be $RCD(K, N)$, $\mu, \nu \in \mathscr{P}(X)$ with $\mu \ll \mathfrak{m}$.

Then:

- There is only one optimal plan
- Such plan is induced by a map T
- For μ-a.e. x there is only one geodesic γ^x from x to $T(x)$
- For μ-a.e. $x \neq y$ we have $\gamma^x_t \neq \gamma^y_t$ for every $t \in [0, 1)$

Optimal maps

Thm. (G., Rajala, Sturm '13) Let $(X, \mathsf{d}, \mathfrak{m})$ be $RCD(K, N)$, $\mu, \nu \in \mathscr{P}(X)$ with $\mu \ll \mathfrak{m}$.

Then:

- There is only one optimal plan
- Such plan is induced by a map T
- For μ-a.e. x there is only one geodesic γ^x from x to $T(x)$
- For μ-a.e. $x \neq y$ we have $\gamma^x_t \neq \gamma^y_t$ for every $t \in [0, 1)$

In particular the $RCD(K, N)$ condition can be localized along geodesics, and if $\mu \leq C\mathfrak{m}$, then $\mu_t \leq C'\mathfrak{m}$ for every $t \in [0, \frac{1}{2}]$

Analytic properties of $RCD(K, N)$ spaces

- Differential calculus on infinitesimally Hilbertian spaces
- The heat flow on $RCD(K, \infty)$ spaces again
- Bochner inequality
- Optimal maps
- Distributional Laplacian

Distributional Laplacian

Let $(X, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian and locally compact, $\Omega \subset X$ open, $g \in S^2(\Omega)$

Distributional Laplacian

Let $(X, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian and locally compact, $\Omega \subset X$ open, $g \in S^2(\Omega)$

We say that $g \in D(\Delta, \Omega)$ if there exists a Radon measure μ on Ω such that

$$-\int_\Omega \nabla f \cdot \nabla g \, \mathrm{d}\mathfrak{m} = \int_\Omega f \, \mathrm{d}\mu,$$

holds for every f Lipschitz with $\operatorname{supp}(f) \subset\subset \Omega$.

In this case we put $\Delta g_{|_\Omega} := \mu$

Calculus rules

Linearity

$$\Delta(\alpha_1 g_1 + \alpha_2 g_2) = \Delta g_1 + \Delta g_2$$

Chain rule

$$\Delta(\varphi \circ g) = \varphi' \circ g \, \Delta g + \varphi'' \circ g |Dg|^2 \mathfrak{m}$$

Leibniz rule

$$\Delta(g_1 g_2) = g_1 \Delta g_2 + g_2 \Delta g_1 + 2 \nabla g_1 \cdot \nabla g_2 \mathfrak{m}$$

Relations with nonlinear potential theory

Theorem (G. '12. G. Mondino '12) Let $(X, \mathsf{d}, \mathfrak{m})$ be inf. Hilb., with doubling measure and supporting a 2-Poincaré inequality.
Let $\Omega \subset X$ and $g \in S^2(\Omega)$.

Relations with nonlinear potential theory

Theorem (G. '12. G. Mondino '12) Let $(X, \mathsf{d}, \mathfrak{m})$ be inf. Hilb., with doubling measure and supporting a 2-Poincaré inequality.
Let $\Omega \subset X$ and $g \in S^2(\Omega)$.
Then TFAE:

- $g \in D(\Delta, \Omega)$ and $\Delta g \leq 0$
- For every Lipschitz $f \geq 0$ with $\mathrm{supp}(f) \subset\subset \Omega$ we have

$$\int_\Omega |Dg|^2 \,\mathrm{d}\mathfrak{m} \leq \int_\Omega |D(g+f)|^2 \,\mathrm{d}\mathfrak{m}$$

Laplacian comparison

On a Riemannian manifold M with $Ric \geq 0$, $\dim \leq N$ it holds

$$\Delta \frac{1}{2} \mathsf{d}^2(\cdot, \overline{x}) \leq N$$

in the sense of distributions.

Laplacian comparison

On a Riemannian manifold M with $Ric \geq 0$, $\dim \leq N$ it holds

$$\Delta \frac{1}{2} \mathsf{d}^2(\cdot, \overline{x}) \leq N$$

in the sense of distributions.

The same holds on $RCD(0, N)$ spaces:
Thm (G. '12) For $(X, \mathsf{d}, \mathfrak{m})$ $RCD(0, N)$ and $\overline{x} \in X$ we have

$$\Delta \frac{\mathsf{d}^2(\cdot, \overline{x})}{2} \leq N\mathfrak{m}$$

Idea of the proof (1/2)

Pick $f \geq 0$ Lipschitz with compact support and let $\rho := cf^{\frac{N}{N-1}}$

$$\mu_0 := \rho\mathfrak{m}, \qquad \mu_1 := \delta_{\overline{x}}, \qquad t \mapsto \mu_t \text{ the geodesic connecting them}$$

The geodesic convexity of $\mathcal{U}_N$ gives

$$\varlimsup_{t \downarrow 0} \frac{\mathcal{U}_N(\mu_t) - \mathcal{U}_N(\mu_0)}{t} \leq \mathcal{U}_N(\mu_1) - \mathcal{U}_N(\mu_0) = c^{1-\frac{1}{N}} \int f \, \mathrm{d}\mathfrak{m}$$

Idea of the proof (2/2)

Let $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ be the lifting of (μ_t) and notice that

$$\begin{aligned} \mathcal{U}_N(\mu_t) - \mathcal{U}_N(\mu_0) &\geq \int u_N'(\rho)\,\mathrm{d}(\mu_t - \mu_0) \\ &= \int u_N'(\rho)(\gamma_t) - u_N'(\rho)(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma) \end{aligned}$$

Idea of the proof (2/2)

Let $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ be the lifting of (μ_t) and notice that

$$\begin{aligned}\mathcal{U}_N(\mu_t) - \mathcal{U}_N(\mu_0) &\geq \int u_N'(\rho)\,\mathrm{d}(\mu_t - \mu_0)\\ &= \int u_N'(\rho)(\gamma_t) - u_N'(\rho)(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\end{aligned}$$

Notice that $\boldsymbol{\pi}$ represents the gradient of $\varphi := -\frac{\mathsf{d}^2(\cdot,\bar{x})}{2}$ to get

$$\begin{aligned}\lim_{t\downarrow 0} \frac{\mathcal{U}_N(\mu_t) - \mathcal{U}_N(\mu_0)}{t} &\geq \int \nabla(u_N'(\rho)) \cdot \nabla\varphi\ (\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &= \frac{c^{1-\frac{1}{N}}}{N} \int \nabla f \cdot \nabla\varphi\,\mathrm{d}\mathfrak{m}\end{aligned}$$

Idea of the proof (2/2)

Let $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ be the lifting of (μ_t) and notice that

$$\begin{aligned}\mathcal{U}_N(\mu_t) - \mathcal{U}_N(\mu_0) &\geq \int u_N'(\rho)\,\mathrm{d}(\mu_t - \mu_0)\\ &= \int u_N'(\rho)(\gamma_t) - u_N'(\rho)(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\end{aligned}$$

Notice that $\boldsymbol{\pi}$ represents the gradient of $\varphi := -\frac{\mathrm{d}^2(\cdot,\bar{x})}{2}$ to get

$$\begin{aligned}\lim_{t\downarrow 0} \frac{\mathcal{U}_N(\mu_t) - \mathcal{U}_N(\mu_0)}{t} &\geq \int \nabla(u_N'(\rho)) \cdot \nabla\varphi\ (\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &= \frac{c^{1-\frac{1}{N}}}{N} \int \nabla f \cdot \nabla\varphi\,\mathrm{d}\mathfrak{m}\end{aligned}$$

Hence

$$-\frac{1}{N} \int \nabla f \cdot \nabla \frac{\mathrm{d}^2(\cdot,\bar{x})}{2}\,\mathrm{d}\mathfrak{m} \leq \int f\,\mathrm{d}\mathfrak{m}, \qquad \forall f \geq 0,\ \text{Lip with cpt supp}$$

Thank you