PARTIAL REGULARITY OF OPTIMAL TRANSPORT MAPS (J/W ~~A.~~ Figalli)

ρ_0, ρ_1 prob. density ~~ispt~~ defined on two open sets Ω_0, Ω_1 (or on a Riem. manifold M). $c: X \times Y \to \mathbb{R}$ cost function.

$$\text{(O.T)} \quad \min_{T_\#(\rho_0 dx) = \rho_1 dy} \int_X c(x, T(x))\, \rho_0(x)\, dx$$

~~Associated to OT there exists a dual pr~~

$$\min \left\{ \int_{X \times Y} c(x,y)\, d\gamma(x,y) \quad \begin{array}{l} (\pi_1)_\#(\gamma) = \rho_0\, dx \\ (\pi_2)_\#(\gamma) = \rho_1\, dy \end{array} \right\}$$

spt $\gamma \subseteq$ c-monoton set.

$\Downarrow$

"graph" of $\partial^c u$ for some c-convex function.

$$u = \sup_{y \in \Omega_1} \{ -c(x,y) + \lambda_y \}$$

$$\partial^c u(x) = \{ y : u(z) + c(z,y) \geq u(x) + c(x,y) \quad \forall z \in \Omega_0 \}$$

If: $\partial^c u(x)$ is univalued for ~~a.e.~~ ρ_0 a.e. x

$\Rightarrow \exists$ optimal T.

1) $c \in C^2(X \times Y)$

2) $y \to \nabla_x c(x,y)$ is inj. ($+ \; x \to \nabla_y c(x,y)$ is inj.)

($+ \det \nabla_{xy} c \neq 0$)

$\boxed{\text{OR} \quad c = d^2/2 \text{ on a Riem. manifold.}}$

$z \to$ ~~c(z,z)~~ $u(z) + c(z,y)$ has a min. ~~at x~~ iff.

a) $\nabla u(x) = -\nabla_x c(x,y)$

b) $\nabla^2 u + \nabla_{xx} c(x,y) \geq 0$

a) $\Rightarrow$ $y = (\nabla_x c(x,\cdot))^{-1}(-\nabla u(x)) = c\text{-}\exp_x(-\nabla u(x))$

Twist.

(in case $c = d^2/2$

$c\text{-}\exp = \exp$!)

$+$

Bi - Twist.

$T_\#(\rho_0) = \rho_1 \Rightarrow |\det \nabla T| = \dfrac{\rho_0}{\rho_1 \circ T}$ | ~~$\nabla_x c(x,T(x)) = \nabla u(x) = 0$~~

$\nabla u(x) + \nabla_x c(x,T(x)) = 0$

$\nabla_{xx} u(x) + \nabla_{xx} c(x,T(x)) = \nabla_{xy} c \; \nabla_x T$

$\det(\nabla_{xx} u + \nabla_{xx} c(x,T(x))) = |\det \nabla_{xy} c| \cdot \dfrac{\rho_0}{\rho_1 \circ T}$

≥ 0

by (b)

The eq. is elliptic on c- convex. function:

<u>Very interesting special case</u>: $c(x,y) = \dfrac{|x-y|^2}{2}$ in $\mathbb{R}^n$

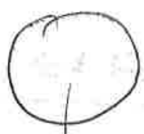
$\dfrac{|x-y|^2}{2} = \underbrace{\tfrac{1}{2}|x|^2 + \tfrac{1}{2}|y|^2}_{\text{"Null Lagrangian terms"}} - x \cdot y \to f = c \Rightarrow u$ is convex

$\det(\nabla^2 u) = \rho_0 / \ldots$

What is known up to know, and how we prove it?

Q: - In which sense is the M.A. equation satisfied?

- Can this give regularity?



$\rho_0 = 1_B$

B^+ B^-

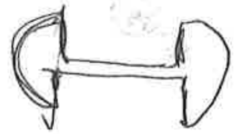
$\rho_1 = 1_{B^+ \cup B^-}$

$c(x,y) = \frac{|x-y|^2}{2}$

~~PROOF~~. T cannot be continuos.

$T = \nabla\left(|x|^2/2 + |x_1| \right)$

similar ex if



Transport solutions to the M.A. equations are not ~~"viscosity solutions" (exactly)~~ "Alexandrov solutions":

~~Alexandrov~~ $\mu_u(A) = \left| \bigcup_{x \in A} \partial u(x) \right|$

Ale solut. $\Leftrightarrow$ $\mu_u = \frac{\rho_0}{\rho_1 \circ \nabla u} dx$ (← I am imposing no singular part)

in the ex. $\mu_u = \frac{\rho_0}{\rho_1 \circ \nabla u} dx + \mathcal{H}^1 \llcorner \{y\text{-axis}\}$

CAFFARELLI: Transport solutions are Alek solutions if Ω_1 (the target) is CONVEX

STRATEGY:

Convex target

$\downarrow$

Alex. solut.

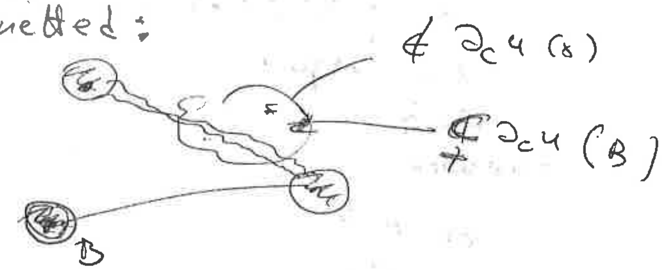
$\downarrow$

Regularity for Alex. sol. $\begin{cases} \frac{1}{\lambda} \le \rho_0, \rho_1 \le \lambda \to T \in C^{0,\beta} \cap W^{1,1+\varepsilon} \\ \rho_0, \rho_1 \in C^{k,\alpha} \to T \in C^{k+1,\alpha} \end{cases}$

GENERAL COST:

An additional problem:

$\partial_c u(x)$ can be disconnected:



$\to$ Counter-examples.

M.T.W. condition ("$\Rightarrow$" $\partial_c u$ is connected.)

$+$

c-convexity of the target (equivalent to the conv in the quadratic case)

$\downarrow$

Alex. sol.

$\downarrow$

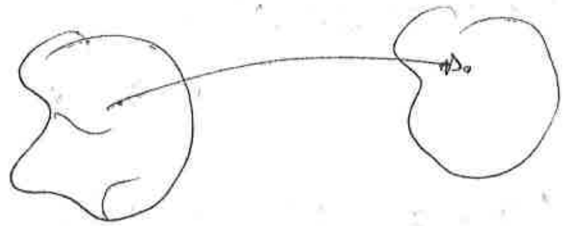
Reg. for Alex.-solut. $\to$ $\begin{cases} \frac{1}{\lambda} \le \rho_0, \rho_1 \le \lambda \quad T \in C^{0,\beta} \cap W^{1,1+\varepsilon} \\ \rho_0, \rho_1 \in C^{k,\alpha} \to T \in C^{k+1,\alpha} \end{cases}$

What happens if the target is not convex (c-convex ?)

THM (Figalli-Kim) quadratic case. $\exists\ \Sigma_0, \Sigma$ closed $|\Sigma_0| = |\Sigma_1| = 0$

$T: \Omega_0 \setminus \Sigma_0 \to \Omega_1 \setminus \Sigma_1$ is ~~smooth~~ homeo (smooth).

Idea



$\{x : \partial u(x) \subseteq \Omega_1\}$ is open of full measure. and u is Alex solut.

REASONABLE

THM (?) (?) If c-sat. MTW.

probably the same.

In $n=2$ "description" of singular set.

~~GENE~~
RMK: In case $c = d^2/2$ MTW gives very strong geometric conditions on the Manifold: (ex $K_M \geq 0$ every-where)

— S^n, quot., perturb. ($\mathbb{RP}^n$), $\mathbb{T}^n$, $\mathbb{CP}^n$

+ , product,

THM (D-Figalli) For a GENERAL C^2-cost satisfying BI-TWIST + ($+\det D_{xy}c \neq 0$)

or

for $c = d^2/2$ on a GENERIC RIEMANNIAN manifold, $\forall$ ρ_0, ρ_1 continuous

$\exists$ Σ_0, Σ_1 closed set $|\Sigma_0| = |\Sigma_1| = 0$

s.t. $T: \Omega_0 \setminus \Sigma_0 \longrightarrow \Omega_1 \setminus \Sigma_1$ is a C^{β}_{loc} homeo

$\forall$ $\beta < 1$

In addition if $c \in C^{k+2,\alpha}$, $\rho_0, \rho_1 \in C^{k,\alpha} \Rightarrow T \in C^{k+1,\alpha}_{loc}$

$\uparrow$

"ε - regularity THM" (similar argument in CAFFARELLI DAL MAR-NGUYEN.)

LEMMA: $\forall \beta$ $\exists$ δ_0, η_0 s.t. if:

a) $\|c(x,y) + x\cdot y\|_{C^0(B_1)} + \|\rho_0 - 1\|_{C^0(B_1)} + \|\rho_1 - 1\|_{C^0(B_1)} \le \delta_0$

b) $T(B_1) \simeq B_1$

$$\|u - |x|^2/2\|_{C^0(B_1)} \le \eta_0$$

$$\Downarrow$$

$$u \in C^{1,\beta} \text{ in } B_{1/2}$$

RMK: After dilation and change of variable
a) holds at every point

b) holds at every TWICE DIFFERENTIABILITY POINT for the Kantorovich potential u
i.e. (b) holds a.e. almost everywhere

Idea of the Proof of the Lemma:

By (a) $\Rightarrow$ $\|u-v\|_{C^0} \le \omega(\delta_0)$ for

where v is an Alek. solution of

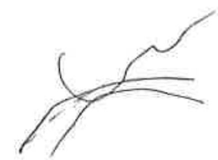
$\det D^2 v = 1$ (this part is by contradiction compactness and is ~~based~~ a "transport proof")

by (b) v enjoys universal estimates

$$Id/K \le D^2 v = A^{-2} \le K\, Id \quad \exists\, K \text{ univ.}$$

$$S_u(h) = \{z : u(z) \le u(x) + c(x,y) - c(z,y) + h\}$$

$$S_v(h) = \{z : v(z) \le v(x) + y\cdot(z-x) + h\}$$



$\delta_0 \ll h_0 \ll 1$

$$S_u(h_0) \simeq S_v(h_0) \simeq A(B_{\sqrt{h_0}})$$

$$T(S_u(h_0)) \simeq \nabla v(S_v(h_0)) \simeq A^{-1}(B_{\sqrt{h_0}})$$

$$\left\| u - \frac{|A^{-1}x|^2}{2} \right\|_{C^0(A(B_{\sqrt{h_0}}))} \le$$

$$\|u-v\|_{C^0(A(B_{\sqrt{h_0}}))} + \|v - D^2v\, x,x\|_{C^0(A(B_{\sqrt{h_0}}))} \lesssim$$

"Here I can put first order terms."

$$\lesssim \omega(\delta_0) + (\sqrt{h_0})^{3/2} \le \eta_0\, h_0$$

$\delta_0 \ll h_0 \ll 1$

$$\tilde{x} = \frac{1}{\sqrt{h_0}} A^{-1} x \in B_1$$

$$\tilde{y} = \frac{1}{\sqrt{h_0}} A x \in B_1$$

$$\tilde{u}(\tilde{x}) = \frac{1}{h} u(x)$$

$$\tilde{c}(\tilde{x}, \tilde{y}) = \frac{1}{h} c(x, y)$$

$$\tilde{\rho}_0(\tilde{x}) = \rho_0(x) \longleftarrow \boxed{\det A = 1}$$

$$\tilde{\rho}_1(\tilde{x}) = \rho_1(x)$$

$$\left\| \tilde{u} - \frac{|\tilde{x}|^2}{2} \right\|_{C^0(B_1)} \le \eta_0$$

$$\|\tilde{c}(\tilde{x}, \tilde{y}) + \tilde{x}\tilde{y}\| + \|\tilde{\rho}_0(\tilde{x}) - 1\| + \|\tilde{\rho}_1(\tilde{x}) - 1\| \le \delta_0$$

I can iterate.

$$\frac{Id}{K} \le A \le K\, Id$$

$\longmapsto$ At the j-th step in the original variables:

$$\left\| u - (K)^{j} \frac{|x|^2}{2} \right\|_{C^0\left(B_{\left(\frac{\sqrt{h_0}}{K}\right)^{j}}\right)} \le h_0^{\,j}$$

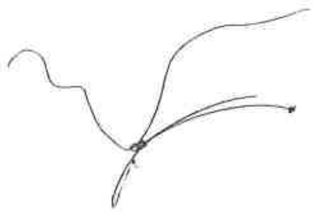
$$\|u\| \lesssim K^{2j} \rho_0^{2j} \underset{B_{\rho_0^j}}{} \;\; \rho_0^{j} \lesssim (\rho_0^{j})^{1+\beta}$$

$$\rho_0 << 1$$

$\Rightarrow u \in C^{1,\beta}$ ~~$\Rightarrow$ a viscosity solution~~

$\partial_i u(x) = C$ - exp d ... $\to$ disconnected

$u \in C^{1,\beta}$



y

$\partial_c u(x) = c - \exp(-\nabla u(x))$ ← 1 point (connected ?)

⇓

u is a "viscosity" solut.

⇓

comparison allows to quantify $\delta_0 \to w(\delta_0) \approx \delta_0^{\gamma}$

⇓

Same argument gives $u \in C^{2,\alpha'}$ (If $\rho_0, \rho_1, \partial_c^2 \in C^{0,\alpha}$) ⇒ classical solution of a <u>uniformly</u> elliptic equation. ⇒ <u>analytic</u>