$\Phi: S^1 = \mathbb{R}/\mathbb{Z} \circlearrowleft$ $\quad$ $\Phi \in C^1$; $\Phi' > 1$.

$\Phi_2 : x \longmapsto 2x \mod 1$

$\Phi_d : x \longmapsto dx \mod 1$.

Invariant measure.

i.e. Lebesgue λ is invariant ~~under~~ Φ_d

more generally any expanding C^1 Φ has a unique a.c. inv. μ.

e.g. Φ_2 : other invariant measure
Bernoulli

Furstenberg conjecture

If $\mu \in \mathcal{P}(S^1)$ is atomless, invariant for Φ_2 and Φ_3

then $\mu = \lambda$.

action on measure $\Phi_\# : \mathcal{P}(S^1) \circlearrowleft$

inv. measure of Φ as fixed point of $\Phi_\#$

Differential structure on $\mathcal{P}(S^1)$.

Wasserstein metric; $W(\mu, \nu) = \left(\inf_{\pi \in \Pi(x,y)} \int d^2(x,y) \, d\pi(x,y) \right)^{\frac{1}{2}}$

Geodesic.

If $(\mu_t)_{0\le t\le 1}$ is a geodesic, (i.e. $W(\mu_t, \mu_s) = c\,|t-s|\ \forall t,s$)

then $\exists\, \mu \in P(G(S'))$ where $G(S')$ is the set of geodesic of S'

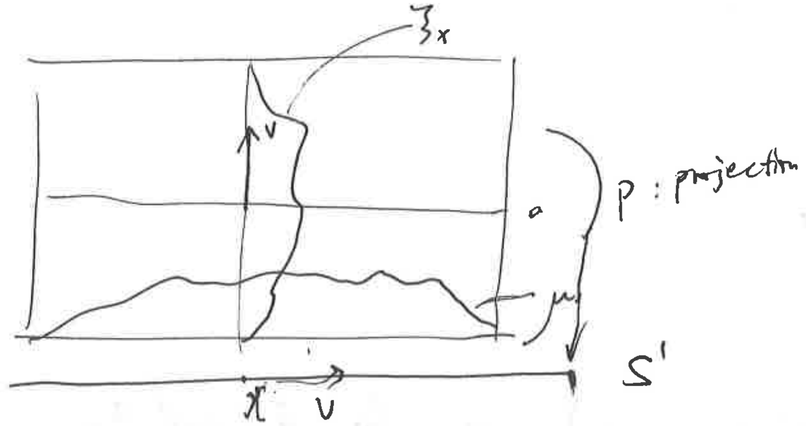
s.t. $e_{t\#}\mu = \mu_t$ where $e_t : G(S') \longrightarrow S'$, $\gamma \longmapsto \gamma_t$.

$(e_0, e_1)_\#\mu$ is optimal coupling

$$PT_\mu = \{\xi \in P(TS') \mid P_\#\xi = \mu,\ W_\mu(\xi, 0) < \infty\}.$$

$\xi \rightsquigarrow (\xi_x)_{x\in S'}$ s.t.

$$\xi = \int \xi_x\, \mu(dx)$$



$\exp_x(v) = x+v$.

$$W_\mu^2(\xi, \zeta) = \int_{S'} W_R^2(\xi_x, \zeta_x)\, \mu(dx)$$

$h_t : (x, v) \longmapsto (x, tv)$ acts on PT_μ

$t\cdot\xi := h_{t\#}\xi$.

cone structure

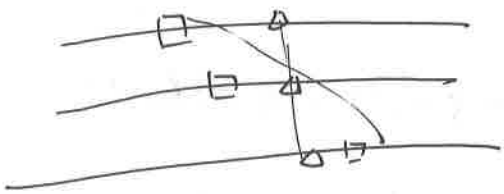
Restangent vector $\rightsquigarrow$ curve

$$\left(\exp_\#(t\cdot\xi)\right)_t.$$

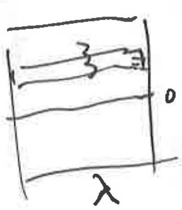
$$W(\mu, \exp_\#(t\cdot\xi)) \le t\, W_\mu(\xi, 0).$$

usually "$\le$" is "$<$":

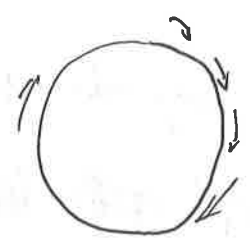
If ξ is not concentrated on a section:



e.g. $\mu = \lambda$, ξ concentrated on $V = ck$.



TS^1



$$T_\mu = \overline{\{\xi \in PT_\mu \mid \exp_\#(t,\xi) \text{ is geodesic of speed } W_\mu(\xi,0) \text{ for small } t\}}^{W_\mu}$$

<u>Thm</u> If μ is without atom then

T_μ is a vector space $\overline{\{\nabla f,\ f \text{ smooth}\}}^{W_\mu}$,

and W_μ induce a ~~def~~ dot product making T_μ <u>Hillbert</u> space.

Remark. if $\mu = \rho\lambda$ with $0 < k \leq \rho \leq K$.

then $T_\mu \simeq L^2_0(\rho\lambda) = \{ V$ vector filds on $S^1 \mid \int V d\lambda = 0 \}$

Identify $\xi \in PT_\gamma$ concentrated on V and V. $\mu + tV := \exp_{p\#}(t.\xi)$

$\Phi_\#$ action near $\rho\lambda$ (a.c.i.m.)

$F : \mathcal{P}(S^1) \circlearrowleft$ is differentiable with derivative L

at μ if $F(\mu + tV) = F(\mu) + t\mathcal{L}(V) + \cancel{st}$ l.o.t.

$$W(F(\mu + tV), F(\mu) + t\mathcal{L}V) = o(t)$$

Thm (K)

Let $\Phi \in C^2$ and expanding $S^1 \circlearrowleft$. with a.c.i.m. $\rho\lambda$. Then $\Phi_\#$ is

differentiable at $\rho\lambda$ with ~~diffe~~ derivative $\mathcal{L}$.

$\mathcal{L}_d := D_\lambda(\Phi_{d\#})$ $\qquad V \mapsto V\left(\frac{\cdot}{d}\right) + V\left(\frac{\cdot + 1}{d}\right) + \cdots + V\left(\frac{\cdots + d - 1}{d}\right)$

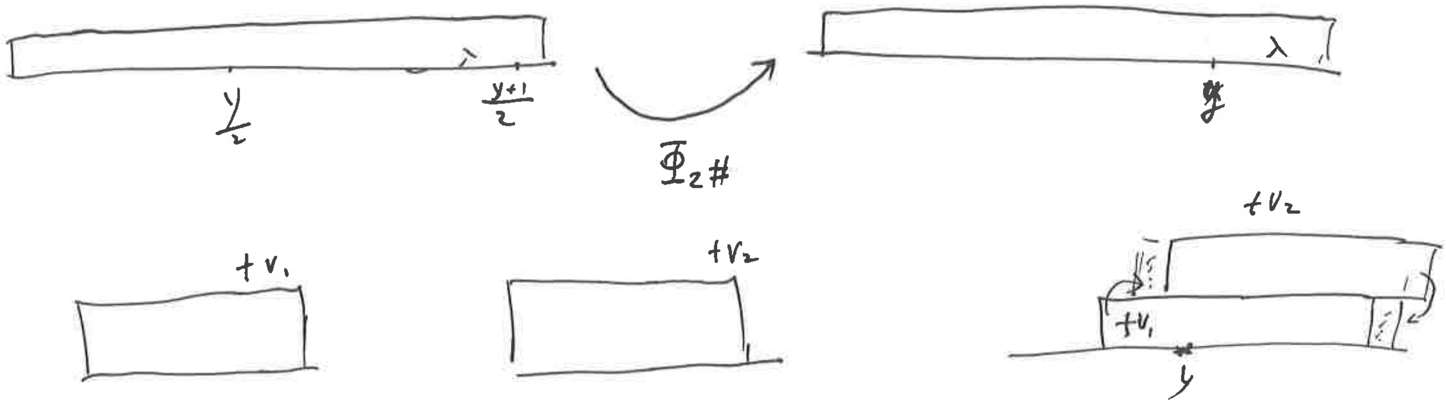
- spectral properties. $\exists R > 1$, s.t $D(0,R) \subseteq \mathbb{C}$ is made of eigenvalue of $\mathcal{L}$ of infinite multiplicity.

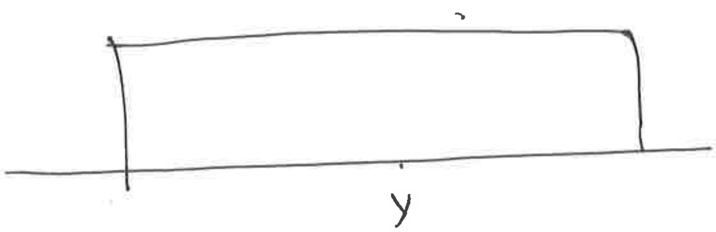
Cor If $E_1(\mathcal{L}_d)$ is the 1-eigenspace of $\mathcal{L}_d$.

Then $E_1(\mathcal{L}_2) \cap E_1(\mathcal{L}_3)$ is infinite-dim.

$$\bigcap_{d \in \mathbb{N}} E_1(\mathcal{L}_d) \text{ is 2-dimensional.}$$

Computation of $\mathcal{L}_2$:





What happens at y.
when looking at

$$\lambda + t\underbrace{\left(v\left(\frac{\cdot}{2}\right)+v\left(\frac{\cdot+1}{2}\right)\right)}_{\mathcal{L}_2 v}$$