Schrödinger problem
Application to discrete metric graphs

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Interpolations in $\mathcal{P}(\mathcal{X})$

- Standard affine interpolation between μ_0 and μ_1
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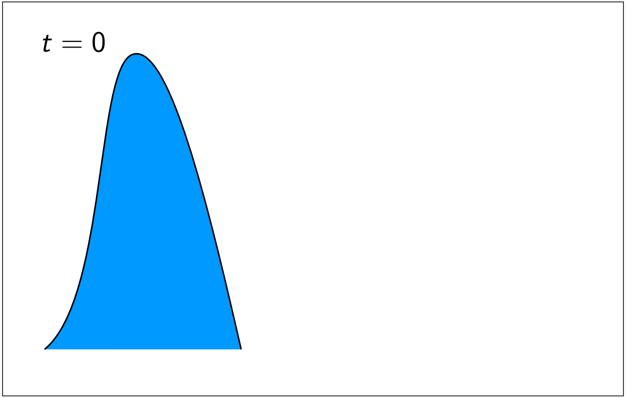

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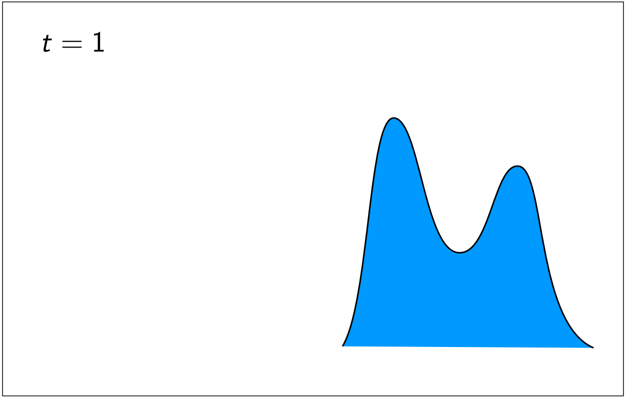

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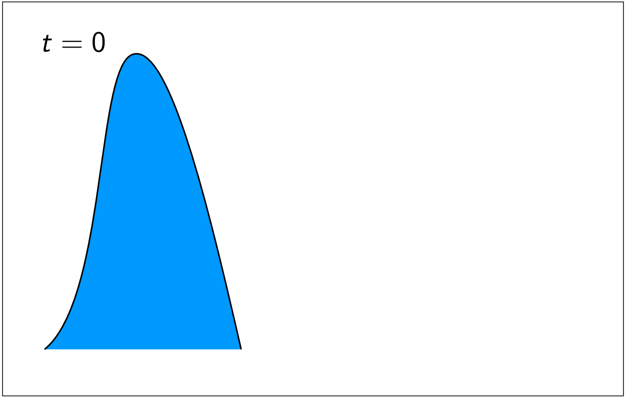

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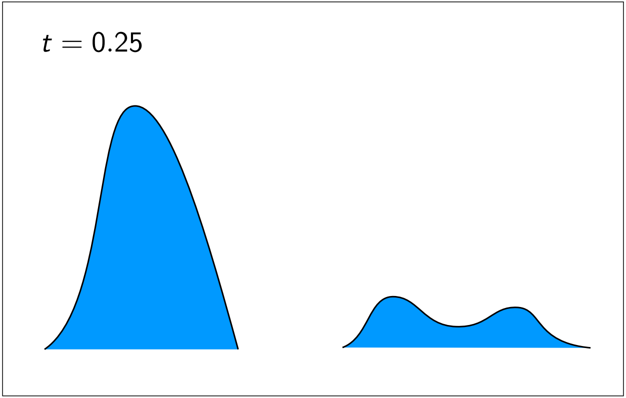

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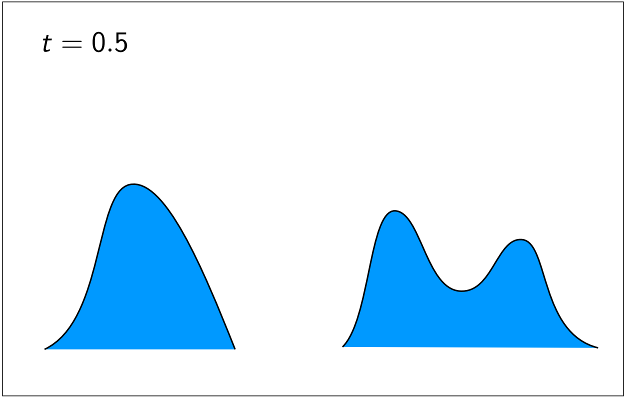

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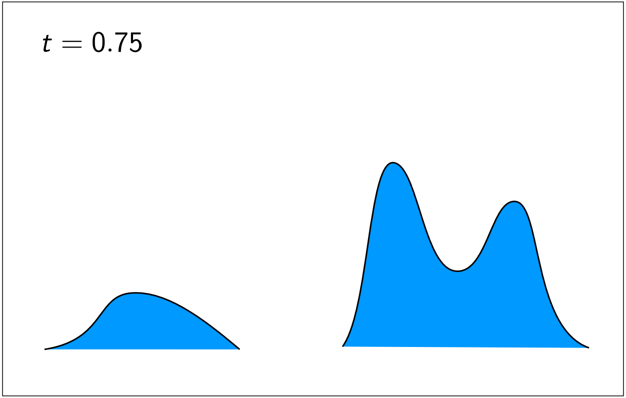

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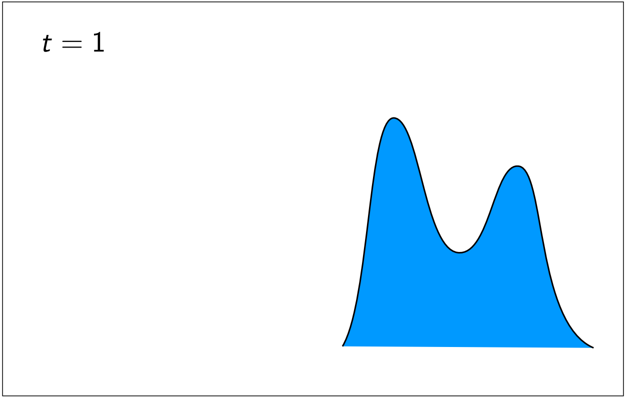

Interpolations in $\mathcal{P}(\mathcal{X})$

Affine interpolations require mass transference with infinite speed

- Denial of the geometry of $\mathcal{X}$
- We need interpolations built upon *trans*-portation, not *tele*-portation

Interpolations in $\mathcal{P}(\mathcal{X})$

- We seek interpolations of this type

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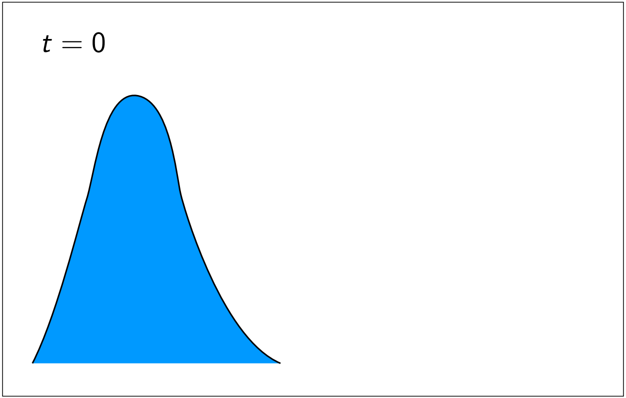

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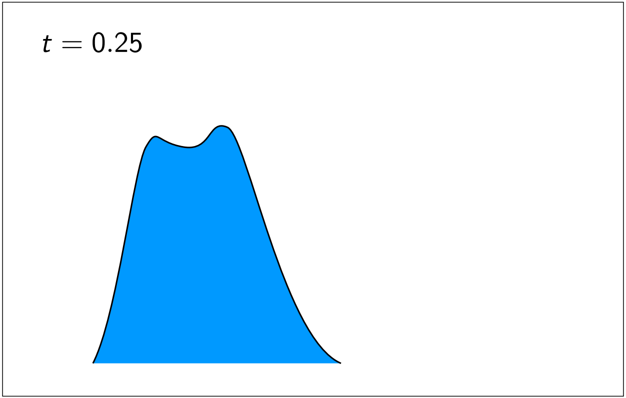

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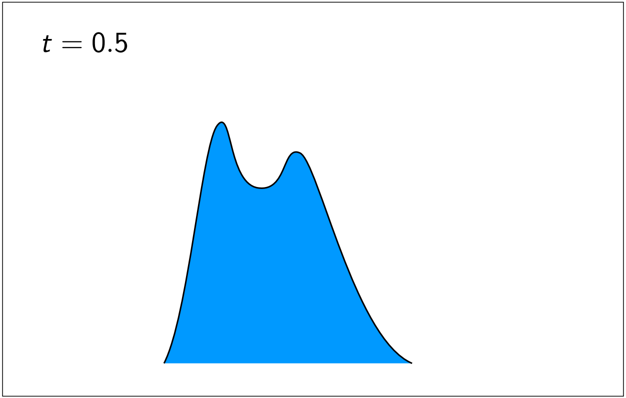

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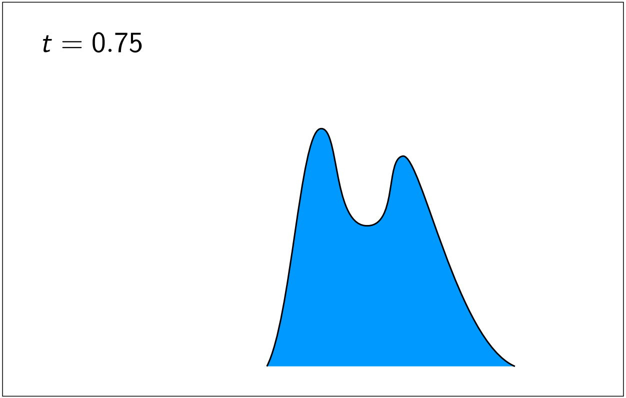

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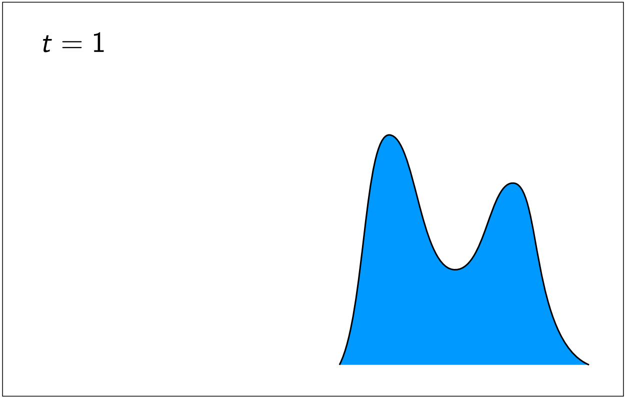


Quadratic Monge-Kantorovich problem

- $(\mathcal{X}, d)$ metric space
- transport cost: $c = d^2$
- $\mu_0, \mu_1 \in \mathcal{P}(\mathcal{X})$ prescribed marginals
- coupling: $\pi \in \mathcal{P}(\mathcal{X}^2)$ s. t. $\begin{cases} \pi_0(dx) & := & \pi(dx \times \mathcal{X}) & = & \mu_0 \\ \pi_1(dy) & := & \pi(\mathcal{X} \times dy) & = & \mu_1 \end{cases}$

Quadratic Monge-Kantorovich problem

$$\int_{\mathcal{X}^2} d^2(x, y)\, \pi(dxdy) \to \min;\ \pi \in \mathcal{P}(\mathcal{X}^2) : \pi_0 = \mu_0, \pi_1 = \mu_1 \qquad \text{(MK)}$$

Wasserstein distance

- $\mathcal{P}_2(\mathcal{X}) := \left\{\mu \in \mathcal{P}(\mathcal{X}) : \int_{\mathcal{X}} d^2(x_o, x)\,\mu(dx) < \infty\right\}$

Theorem

For all $\mu_0, \mu_1 \in \mathcal{P}_2(\mathcal{X})$,

1. (MK) admits a solution
2. $W_2(\mu_0, \mu_1) := \inf\,(\text{MK})^{1/2} < \infty$ is a distance on $\mathcal{P}_2(\mathcal{X})$ (Wasserstein distance)

Quadratic transport, $\mathcal{X} = M$

$$\int_{\mathcal{X}^2} d^2(x,y)\,\pi(dxdy) \to \min; \quad \pi \in \mathcal{P}(\mathcal{X}^2) : \pi_0 = \mu_0, \pi_1 = \mu_1$$

Theorem $(\mathcal{X} = M,\ c = d^2)$ Brenier, McCann

Suppose $\mu_0, \mu_1 \in \mathcal{P}_{2,ac}(\mathcal{X}^2)$. Then, there exist $\psi_0 : \mathcal{X} \to \mathbb{R}$:

- $\pi^T(dxdy) := \mu_0(dx)\delta_{T(x)}(dy)$ is the unique solution of (MK)
- $T(x) = \exp_x(\nabla\psi_0(x)),\ \mu_0$-a.e.

Notation

- $\Omega = \{\text{paths}\} \subset \mathcal{X}^{[0,1]}$
- $\omega = (\omega_t)_{0\le t\le 1} \in \Omega$
- $X_t : \omega \in \Omega \mapsto \omega_t \in \mathcal{X}, \ \ 0 \le t \le 1$ (canonical process)
- $P \in \mathcal{P}(\Omega)$
- $P_t := (X_t)_\# P \in \mathcal{P}(\mathcal{X})$
- $P_{st} = (X_s, X_t)_\# P \in \mathcal{P}(\mathcal{X}^2)$
- P_0 : initial marginal, P_1 : final marginal, P_{01} : endpoint marginal
- $P^{xy} = P(\cdot \mid X_0 = x, X_1 = y) \in \mathcal{P}(\Omega)$: bridge

Disintegration formula

$$P(\cdot) = \int_{\mathcal{X}^2} P^{xy}(\cdot)\, P_{01}(dxdy) \in \mathcal{P}(\Omega)$$

Dynamical quadratic transport, $\mathcal{X} = M$

Geodesic equality

$d^2(x,y) = \inf_\omega \int_0^1 |\dot\omega_t|^2\,dt; \qquad \omega = (\omega_t)_{0\le t\le 1}:\ \omega_0 = x, \omega_1 = y$

Minimizer: constant speed geodesic γ^{xy}

Dynamical Monge-Kantorovich problem

$$\int_\Omega dP \int_0^1 |\dot X_t|^2\,dt \to \min; \qquad P \in \mathcal{P}(\Omega) : P_0 = \mu_0, P_1 = \mu_1 \qquad (\mathrm{MK_{dyn}})$$

Theorem

1. P solution iff $\quad P(\cdot) = \int_{\mathcal{X}^2} \delta_{\gamma^{xy}}(\cdot)\,\pi(dxdy)\quad$ where π solves $(\mathrm{MK})_2$
2. $\int_\Omega dP \int_0^1 |\dot X_t|^2\,dt = W_2^2(\mu_0,\mu_1)$

Proof. $\int_\Omega dQ \int_0^1 |\dot X_t|^2\,dt = \int_{\mathcal{X}^2}[\int_\Omega dQ^{xy} \int_0^1 |\dot X_t|^2\,dt]\,Q_{01}(dxdy)$
$\ge \int_{\mathcal{X}^2} d^2(x,y)\,Q_{01}(dxdy) \ge \int_{\mathcal{X}^2} d^2(x,y)\,\pi(dxdy) = W_2^2(\mu_0,\mu_1) \quad \square$

Displacement interpolation, $\mathcal{X} = M$

When $\mu_0 \ll \mathrm{vol}$, there is a *unique* solution

$$P(\cdot) = \int_{\mathcal{X}^2} \delta_{\gamma^{xy}}(\cdot)\, \pi(dxdy)$$

Displacement interpolation (McCann)

$[\mu_0, \mu_1]^{\mathrm{disp}} = (\mu_t)_{0 \le t \le 1}$ where $\mu_t := P_t \in \mathcal{P}(\mathcal{X})$

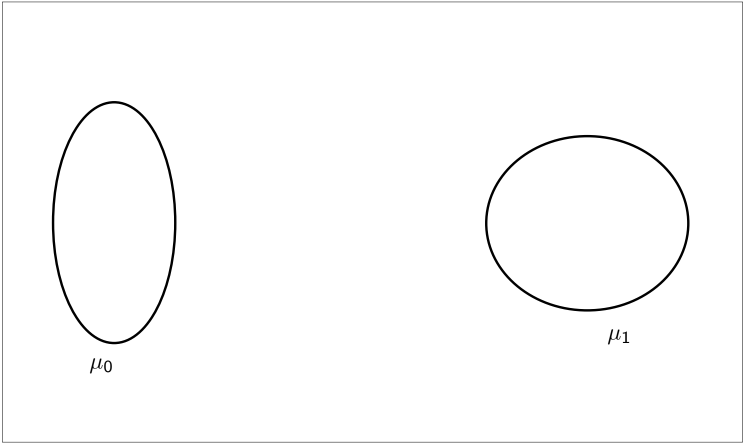
$$\mu_t = \int_{\mathcal{X}^2} \delta_{\gamma_t^{xy}}(\cdot)\, \pi(dxdy) \in \mathcal{P}(\mathcal{X})$$

Proposition

For all $0 \le s, t \le 1$, P_{st} is an optimal coupling of (μ_s, μ_t)

Proof. Otherwise, P wouldn't be optimal $\square$

Displacement interpolation, $\mathcal{X} = M$



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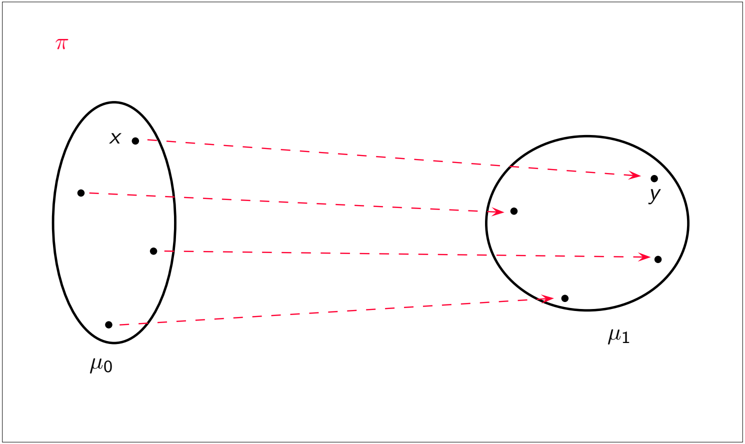

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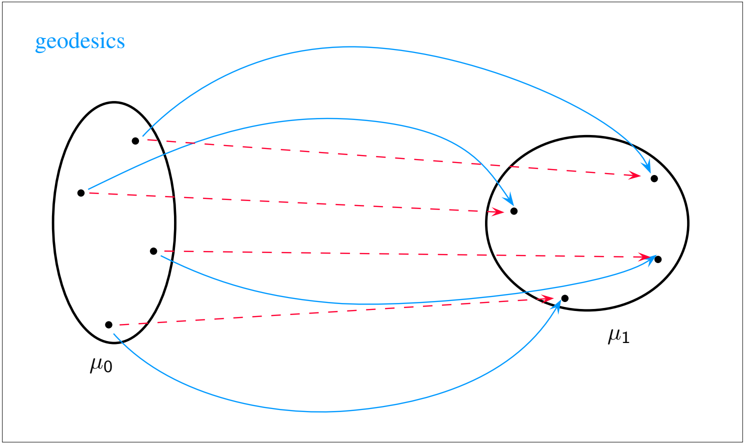

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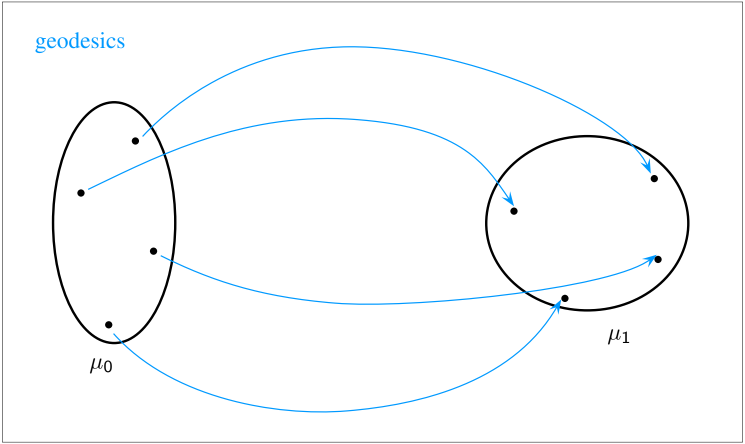

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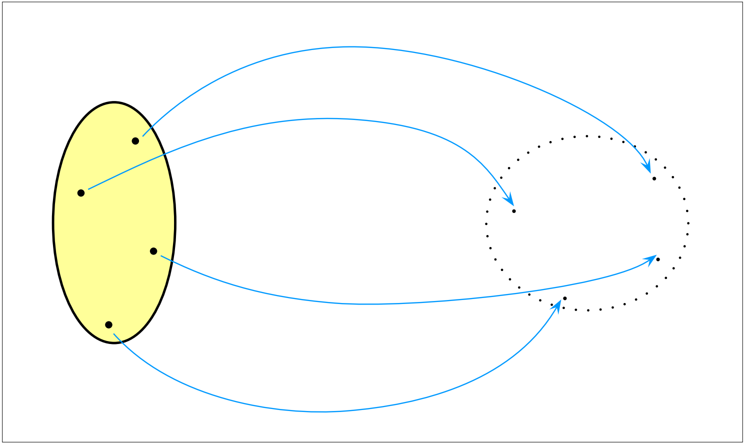

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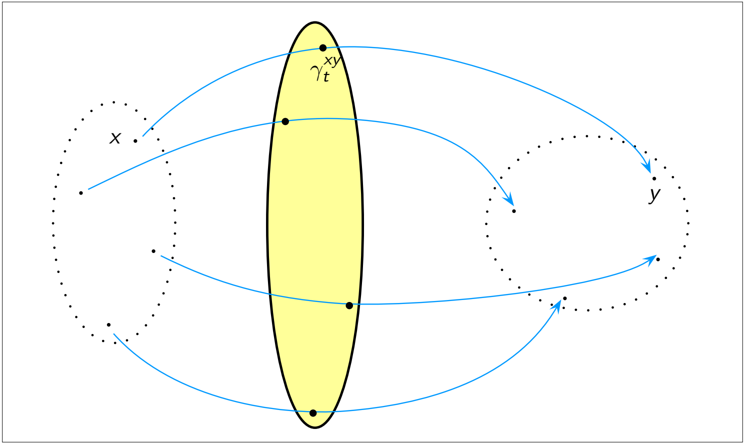

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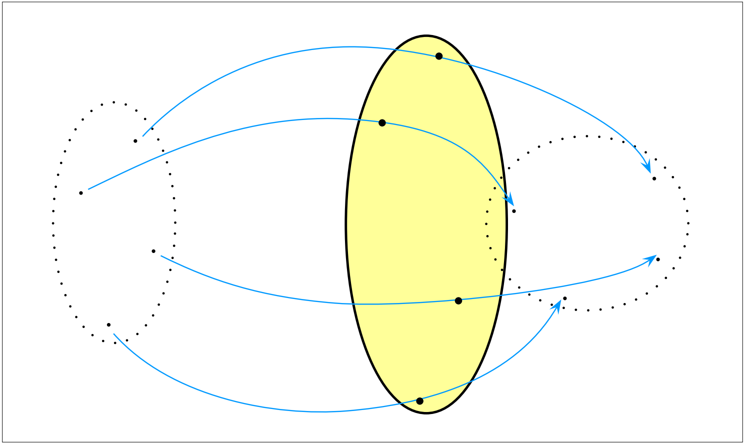

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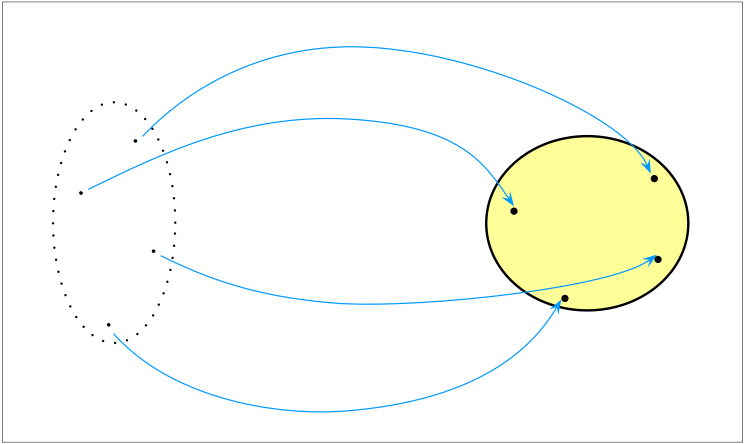


Displacement interpolation, $\mathcal{X} = M$

Dynamical Monge-Kantorovich problem

$$\int_\Omega dP \int_0^1 |\dot X_t|^2 \, dt \to \min; \qquad P \in \mathcal{P}(\Omega) : P_0 = \mu_0, P_1 = \mu_1 \qquad (\mathrm{MK_{dyn}})$$

- $P = \int_{\mathcal{X}^2} \delta_{\gamma^{xy}} \, \pi(dxdy)$
- for some function $\psi : [0,1] \times \mathcal{X} \to \mathbb{R}$, $\quad \dot X_t = \nabla \psi_t(X_t), \quad P\text{-a.e.}$

Benamou-Brenier formula

- $$W_2^2(\mu_0, \mu_1) = \inf_{(\nu, v)} \int_0^1 \langle |v_t|^2, \nu_t \rangle \, dt$$
  $$(\nu, v) : \quad \partial_t \nu + \nabla \cdot (\nu v) = 0, \quad \nu_{|t=0} = \mu_0, \nu_{|t=1} = \mu_1$$
- infimum attained at $(\nu, v) = (\mu, \nabla \psi)$

Displacement interpolation, $\mathcal{X} = M$

Otto's formal calculus

$(\mathcal{P}_2(\mathcal{X}), W_2)$ looks like a Riemannian manifold
Its constant speed geodesics are the displacement interpolations

Displacement interpolation, $\mathcal{X} = M$

Relative entropy

$$H(p|r) := \int \log(dp/dr)\, dp,$$
p : probability measure, r : reference σ-finite measure

- $m = e^{-V}\,\mathrm{vol}$

Convexity of the entropy along displacement interpolations

The following assertions are equivalent

- Hess V + Ric $\geq K$
- for any $[\mu_0, \mu_1]^{\mathrm{disp}}$, $t \in [0,1] \mapsto H(\mu_t|m)$ is K-convex on $(\mathcal{P}_2(\mathcal{X}), W_2)$

- Sturm-von Renesse (2004)
- $H(\mu_t|m) \leq (1-t)H(\mu_0|m) + tH(\mu_1|m)$
  $\qquad -KW_2^2(\mu_0,\mu_1)t(1-t)/2, \qquad 0 \leq t \leq 1$
- non-smooth formulation of $\frac{d^2}{dt^2}H(\mu_t|m) \geq KW_2^2(\mu_0,\mu_1)$

Analytic consequences

When $K > 0$:

- m satisfies an entropy-entropy production (logarithmic Sobolev) inequality
- heat flows convergence exponentially fast to m as $t \to \infty$
- m satisfies a transport-entropy inequality (Talagrand)
- concentration of the measure m

Extension: $(M, d_g, \mathrm{vol}_g) \quad \to \quad (\mathcal{X}, d, m)$

- $(\mathcal{X}, d, m)$ metric measure space
- d allows for
  - stating $(\mathrm{MK})_2$
  - constructing displacement interpolations on $(\mathcal{P}_2(\mathcal{X}), W_2)$
- m allows for defining $H(\cdot|m)$

Definition of curvature lower bound (Lott-Sturm-Villani)

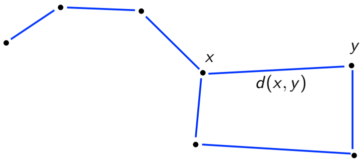
Framework: $(\mathcal{X}, d)$ is a geodesic space
$(\mathcal{X}, d, m)$ has curvature lower bound K if along any displacement interpolation $[\mu_0, \mu_1]^{\mathrm{disp}}$, $t \mapsto H(\mu_t|m)$ is K-convex on $(\mathcal{P}_2(\mathcal{X}), W_2)$

☺ works with
- m-Gromov-Hausdorff limits of Riemannian manifolds
- spaces where $|\nabla u|^2$ has "some meaning" (Ambrosio-Gigli-Savaré)

☹ fails with discrete graphs
- no *regular* geodesics
- "branching"

What is a regular geodesic on a discrete metric graph?

- non-directed discrete metric graph $(\mathcal{X}, \sim, d)$



$x \sim y$ means that (x, y) is an edge

Problem

How to define a constant speed geodesic on $(\mathcal{X}, \sim)$?

- one *must* introduce a random walk on $(\mathcal{X}, \sim)$: $\quad \mathcal{X} \dashrightarrow \mathcal{P}(\mathcal{X})$
- find a good notion of regular geodesic $[\mu_0, \mu_1] = (\mu_t)_{0 \le t \le 1}$ on $\mathcal{P}(\mathcal{X})$
- identify $[\delta_x, \delta_y]$ with the geodesic from x to y

It will also allow to compute $\quad \dfrac{d^2}{dt^2} H(\mu_t | m)$

What is a regular geodesic on a discrete metric graph?

We look for a displacement interpolation on a discrete graph

Several candidates

- h-approximate t-midpoint interpolations (typically $h = 1$)
  $d(x_0, x_t) \leq td(x_0, x_1) + h, \quad d(x_t, x_1) \leq (1-t)d(x_0, x_1) + h$
  Bonciocat-Sturm (2009)
- Maas-Mielke gradient flows: the evolution of a reversible random walk is the gradient flow of an entropy for some distance $\mathcal{W}$ on $\mathcal{P}(\mathcal{X})$
  Maas (2011), Mielke (2011)
- Binomial interpolations
  geodesic: $(x := z_0, z_1, \dots, z_{d(x,y)} =: y), \quad \mu_t(z_k) = \mathcal{B}(d(x,y), t)(k)$
  Johnson (2007), Gozlan-Roberto-Samson-Tetali (2012)

Schrödinger's hot gas experiment

Consider a huge collection of non-interacting identical particles in a heat bath. If the density profile of the system at time $t = 0$ is approximately $\mu_0 \in \mathcal{P}(\mathbb{R}^3)$, you expect it to evolve along the heat flow:

$$\begin{cases} \nu_t = \nu_0 e^{t\Delta/2}, & 0 \leq t \leq 1 \\ \nu_0 = \mu_0 \end{cases}$$

where Δ is the Laplace operator.

Suppose that you observe the density profile of the system at time $t = 1$ to be approximately $\mu_1 \in \mathcal{P}(\mathbb{R}^3)$ with μ_1 *different from the expected* ν_1. Probability of this rare event $\simeq \exp(-CN_{\mathrm{Avogadro}})$.

Schrödinger's question (1931)

Conditionally on this very rare event, what is the *most likely path* $(\mu_t)_{0 \leq t \leq 1} \in \mathcal{P}(\mathbb{R}^3)^{[0,1]}$ of the evolving profile of the particle system?

Reference path measure

The reference path measure describes the random behavior of an individual particle

- $R \in \mathcal{M}_+(\Omega), \quad m \in \mathcal{M}_+(\mathcal{X})$
- R is m-reversible Markov with generator L

Example

R : Brownian motion on the Riemannian manifold $\mathcal{X}, \quad m = \mathrm{vol}$
$L = \Delta/2$

Example

R : simple random walk on the graph $(\mathcal{X}, \sim), \quad m = \sum_{x \in \mathcal{X}} n_x \delta_x$
$Lu(x) = \sum_{y : y \sim x} [u(y) - u(x)]/n_x$

Schrödinger problem

Relative entropy

$$H(p|r) := \int \log\left(\frac{dp}{dr}\right)\, dp = \int \frac{dp}{dr} \log\left(\frac{dp}{dr}\right)\, dr \in (-\infty, \infty].$$

Schrödinger problem (Föllmer, 1985)

$$H(P|R) \to \min; \quad P \in \mathcal{P}(\Omega) : P_0 = \mu_0, P_1 = \mu_1 \qquad (\mathsf{S}_{\mathrm{dyn}})$$

$\mu_0, \mu_1 \in \mathcal{P}(\mathcal{X})$ are the initial and final prescribed profiles

Definition. *R*-entropic interpolation

$[\mu_0, \mu_1]^R := (P_t)_{0 \le t \le 1}$ with P the unique solution of $(\mathsf{S}_{\mathrm{dyn}})$.

It is the answer to Schrödinger's question

Schrödinger problem

Assume $\mathcal{X}$ is discrete and $R_{01}(x,y) > 0$, $\forall x, y$. Take $\mu_0 = \delta_x, \mu_1 = \delta_y$.

- $P = R^{xy}$
- $[\delta_x, \delta_y]^R = (R^{xy}_t)_{0 \le t \le 1}$

More generally

$$P(\cdot) = \int_{\mathcal{X}^2} R^{xy}(\cdot)\, \pi(dxdy)$$

where $\pi \in \mathcal{P}(\mathcal{X}^2)$ is the unique solution of

$$H(\pi | R_{01}) \to \min; \quad \pi \in \mathcal{P}(\mathcal{X}^2) : \pi_0 = \mu_0, \pi_1 = \mu_1 \qquad \text{(S)}$$

R-entropic interpolation

$$\mu_t(\cdot) = \int_{\mathcal{X}^2} R^{xy}_t(\cdot) \pi(dxdy)$$

Schrödinger problem

Result

Assume: R is m-reversible Markov,
$R_{01} \ll m \otimes m, \quad H(\mu_0|m), H(\mu_1|m) < \infty, \dots$

- (S) admits a unique solution $P \in \mathcal{P}(\Omega)$
- There exist f_0, g_1 *measurable* such that

$$P = f_0(X_0) g_1(X_1)\, R$$

Long history: Schrödinger, Bernstein, Fortet, Beurling, Jamison, Csiszár, Rüschendorf-Thomsen, Föllmer-Gantert, L.

Conversely:

If $\begin{cases} P = f_0(X_0) g_1(X_1)\, R \\ H(P \mid R) < \infty \end{cases}$, then P solves (S) with $\begin{cases} \mu_0 = P_0 \\ \mu_1 = P_1 \end{cases}$

Entropic interpolation. Dynamics

$P = f_0(X_0) g_1(X_1)\, R, \qquad \mu_t := P_t$

Theorem (Analogue of Born's formula)

$$\rho_t := \frac{d\mu_t}{dm} = f_t g_t, \quad 0 \le t \le 1$$

$$\begin{cases} f_t(z) & := & E_R[f_0(X_0)|X_t = z] \\ g_t(z) & := & E_R[g_1(X_1)|X_t = z] \end{cases}$$

Equations of motion

1. (f_0, g_1) solves $\begin{cases} f_0 g_0 & = & \rho_0 \\ f_1 g_1 & = & \rho_1 \end{cases}$
2. $\begin{cases} (-\partial_t + L) f = 0 \\ f_{|t=0} = f_0 \end{cases} \qquad \begin{cases} (\partial_t + L) g = 0 \\ g_{|t=1} = g_1 \end{cases}$

Biblio: Schrödinger (1932), Zambrini (1986), L.

Entropic interpolation. Dynamics

1. Pick $\mu_0 = \rho_0\, m, \quad \mu_1 = \rho_1\, m \quad \in \mathcal{P}(\mathcal{X})$
2. Find (f_0, g_1) such that $\begin{cases} f_0 g_0 = \rho_0 \\ \downarrow \uparrow \\ f_1 g_1 = \rho_1 \end{cases}$
3. Solve $\begin{cases} (-\partial_t + L) f = 0 \\ f_{|t=0} = f_0 \end{cases} \qquad \begin{cases} (\partial_t + L) g = 0 \\ g_{|t=1} = g_1 \end{cases}$

Proposition

An alternate definition:

$$\mu_t := f_t g_t\, m, \qquad 0 \le t \le 1$$

is the *R-entropic interpolation* $[\mu_0, \mu_1]^R$.

Entropic interpolation. Dynamics

- $P = f_0(X_0)g_1(X_1)\, R$
- $\begin{cases} f_t(z) & := & E_R[f_0(X_0)|X_t = z] \\ g_t(z) & := & E_R[g_1(X_1)|X_t = z] \end{cases}$
- $\Gamma(u, v) := L(uv) - uLv - vLu$: (carré du champ)

Theorem

- *P is Markov*
- *Generators of P:* $\overleftarrow{A}_t = L + \dfrac{\Gamma(f_t, \cdot)}{f_t}, \quad \overrightarrow{A}_t = L + \dfrac{\Gamma(g_t, \cdot)}{g_t}, \quad 0 \le t \le 1$

Examples

- $\overleftarrow{A}_t = L + \dfrac{\Gamma(f_t, \cdot)}{f_t}, \qquad \overrightarrow{A}_t = L + \dfrac{\Gamma(g_t, \cdot)}{g_t}$
- Define $\begin{cases} \varphi_t(z) & := & \log f_t(z) \\ \psi_t(z) & := & \log g_t(z) \end{cases}$

Example

$$\begin{cases} L = \frac{1}{2}(\Delta - \nabla V \cdot \nabla) \\ R_0 = m = e^{-V}\,\mathrm{vol} \quad \text{(reversing)} \end{cases}, \quad \begin{cases} \overrightarrow{A}_t = L + \nabla\psi_t \cdot \nabla \\ \overleftarrow{A}_t = L + \nabla\varphi_t \cdot \nabla \end{cases}$$

Example

$$\begin{cases} L \to J_x(dy) \\ R_0 = m \quad \text{(reversing)} \end{cases}, \quad \begin{cases} \overrightarrow{A}_t \to \frac{g_t(y)}{g_t(x)}\, J_x(dy) = e^{\psi_t(y)-\psi_t(x)}\, J_x(dy) \\ \overleftarrow{A}_t \to \frac{f_t(y)}{f_t(x)}\, J_x(dy) = e^{\varphi_t(y)-\varphi_t(x)}\, J_x(dy) \end{cases}$$

$Lu(x) = \sum_{y:y\sim x}[u(y) - u(x)]J_x(y) = \int_{\mathcal{X}}[u(y) - u(x)]\, J_x(dy),$
$J_x = \sum_{y:y\sim x} J_x(y)\,\delta_y$

Hamilton-Jacobi-Bellman

$$\begin{cases} \varphi_t(z) & := & \log f_t(z) & = & \log E_R[e^{\varphi_0(X_0)} | X_t = z] \\ \psi_t(z) & := & \log g_t(z) & = & \log E_R[e^{\varphi_1(X_1)} | X_t = z] \end{cases}$$

$$\begin{cases} (-\partial_t + B)\varphi = 0 \\ \varphi_{|t=0} = \varphi_0 \end{cases} \qquad \begin{cases} (\partial_t + B)\psi = 0 \\ \psi_{|t=1} = \psi_1 \end{cases}$$

where

$$Bu := e^{-u} L e^{u}$$

$$Bu(x) = \begin{cases} Lu(x) + |\nabla u|^2(x)/2 \\ Lu(x) + \sum_{y:y\sim x} \tau(u(y) - u(x))\, J_x(y), \quad \tau(a) := e^a - a - 1 \end{cases}$$

Forward-backward

ψ-representation

$$\begin{cases} (\partial_t + B)\psi &= 0; \quad \leftarrow \psi_1 \\ (-\partial_t + \overrightarrow{A}^*_{\psi_t})\mu &= 0; \quad \mu_0 \rightarrow \end{cases}$$

φ-representation

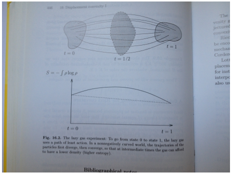
$$\begin{cases} (-\partial_t + B)\varphi &= 0; \quad \varphi_0 \rightarrow \\ (\partial_t + \overleftarrow{A}^*_{\varphi_t})\mu &= 0; \quad \leftarrow \mu_1 \end{cases}$$

Time reversal

$$\varphi + \psi = \log \rho$$

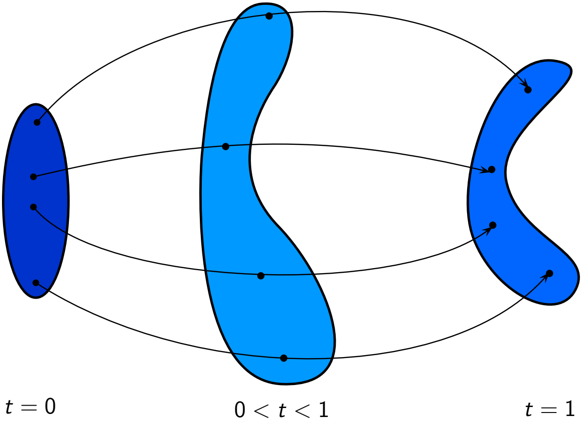
Gas experiments

- Lazy gas experiment
  - Zero temperature
  - Displacement interpolations
  - Optimal transport



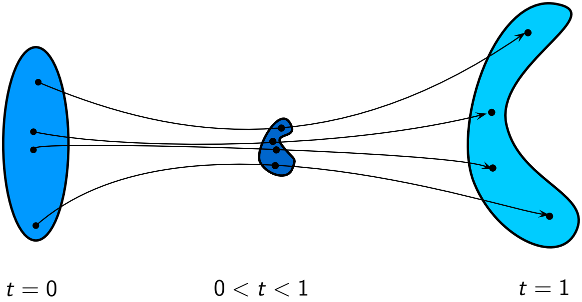
- Not so lazy gas experiment (Schrödinger)
  - Positive temperature
  - Entropic interpolations
  - Minimal entropy

Gas experiments



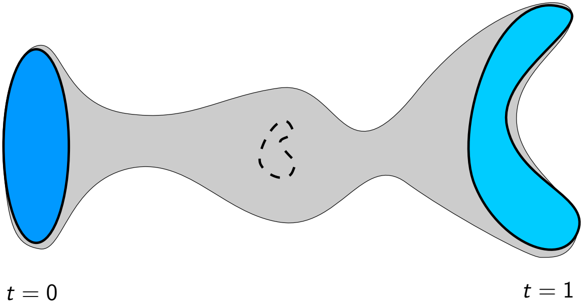
The lazy gas experiment. Positive curvature

Gas experiments



The lazy gas experiment. Negative curvature

Gas experiments



The not so lazy gas experiment in a negative curvature manifold

Convexity of the entropy along entropic interpolations

Fix the interpolation $[\mu_0, \mu_1]$ and consider

$$t \in [0,1] \mapsto H(t) := H(\mu_t | m) \in \mathbb{R}.$$

Define

- $C := B - L$ (nonlinear part of the HJ operator B)
- $\Theta(u) := e^{-u}\Gamma(e^u, u) - Cu$
- $\Theta_2(u) := L\Theta u + e^{-u}\Gamma(e^u, \Theta u) + Bu\, e^{-u}\Gamma(e^u, u) - e^{-u}\Gamma(e^u Bu, u)$

Proposition

- $H'(t) = \langle -\Theta(\varphi_t) + \Theta(\psi_t), \mu_t \rangle$
- $H''(t) = \langle \Theta_2(\varphi_t) + \Theta_2(\psi_t), \mu_t \rangle$

Convexity of the entropy along entropic interpolations

$$H'(t) = \langle -\Theta(\varphi_t) + \Theta(\psi_t), \mu_t \rangle, \qquad H''(t) = \langle \Theta_2(\varphi_t) + \Theta_2(\psi_t), \mu_t \rangle$$

Outline of the proof

$H(t) = \langle \log \rho_t, \mu_t \rangle = \langle \varphi_t + \psi_t, \mu_t \rangle$

Use the rules:

- $\langle u, \dot{\mu}_t \rangle = \begin{cases} \langle -\overleftarrow{A}_{\varphi_t} u, \mu_t \rangle \\ \langle \overrightarrow{A}_{\psi_t} u, \mu_t \rangle \end{cases}$
- $\dot{\varphi} = B\varphi, \quad \dot{\psi} = -B\psi.$

Main feature of this proof

It is essential to take advantage of both directions of time

Convexity of the entropy along entropic interpolations

$$H'(t) = \langle -\Theta(\varphi_t) + \Theta(\psi_t), \mu_t \rangle, \qquad H''(t) = \langle \Theta_2(\varphi_t) + \Theta_2(\psi_t), \mu_t \rangle$$

Example

$L = \frac{1}{2}(\Delta - \nabla V \cdot \nabla)$

- $\Theta(u) = \Gamma(u)/2 = |\nabla u|^2/2$
- $$\begin{aligned} \Theta_2(u) &= L(\Gamma(u)) - 2\Gamma(Lu, u) \\ &:= \Gamma_2(u)/2 \\ &= \big(\|\nabla^2 u\|^2_{\mathrm{HS}} + [\nabla^2 V + \mathrm{Ric}](\nabla u)\big)/2 \end{aligned}$$

- $CD(\kappa, \infty) \overset{\text{def}}{\iff} \Gamma_2 \geq \kappa\Gamma \iff \Theta_2 \geq \kappa\Theta$

Convexity of the entropy along entropic interpolations

$$H'(t) = \langle -\Theta(\varphi_t) + \Theta(\psi_t), \mu_t \rangle, \qquad H''(t) = \langle \Theta_2(\varphi_t) + \Theta_2(\psi_t), \mu_t \rangle$$

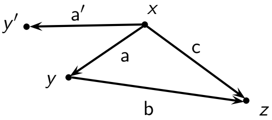
Example

$L \to J_x(dy)$

- $\Theta(u)(x) = \int_{\mathcal{X}} \tau^*(e^{u(y)-u(x)} - 1)\, J_x(dy)$
  $\tau^*(b) = (b+1)\log(b+1) - b$
- $\Theta_2(u) =$ uneasy to interpret

Convexity of the entropy along entropic interpolations

$$\begin{cases} \mathrm{a} & := & Du_x(y) \\ \mathrm{b} & := & Du_y(z) \\ \mathrm{c} & := & Du_x(z) \end{cases}$$



Define $\quad h(\alpha) := \tau^*(e^\alpha - 1) = \alpha e^\alpha - e^\alpha + 1, \quad \alpha \in \mathbb{R}$

Proposition

1. $\Theta u(x) = \sum_{x \to y} h(\mathrm{a})$
2. $$\Theta_2 u(x) = \Big(Bu(x)\Big)^2 + \sum_{x \to y} [J_y(\mathcal{X}) - J_x(\mathcal{X})] h(\mathrm{a}) + \sum_{x \to y \to z} [2e^{\mathrm{a}} h(\mathrm{b}) - h(\mathrm{c})]$$

Remaining problems

- Is it a Bochner formula?
- $k \to \infty$

Recall $\quad \Gamma_2(u) = \|\nabla^2 u\|^2_{\mathrm{HS}} + [\nabla^2 V + \mathrm{Ric}](\nabla u)$

Definitions of local curvature ?

- $H''(t) = \langle \Theta_2(\varphi_t) + \Theta_2(\psi_t), \mu_t \rangle$
- Curvature-dimension bound: $CD(\kappa, \infty) \implies \Theta_2 \geq \kappa\Theta$

Are these definitions relevant?

- The Markov generator L has curvature bounded below by $\kappa \in \mathbb{R}$ if
$$\Theta_2(u)(x) \geq \kappa\Theta(u)(x), \quad \forall u, x.$$
- The curvature at $x \in \mathcal{X}$ of the Markov generator L is
$$\mathrm{curv}_L(x) := \inf_u \frac{\Theta_2(u)}{\Theta(u)}(x)$$

Clue 1

$Lu(x) = [u(x+1) - 2u(x) + u(x-1)]/2$ gives $\mathrm{curv}_L(x) = 0, \quad \forall x \in \mathbb{Z}$

Clue 2

unified proof of logarithmic Sobolev inequality

Cooling the heat bath

Caution

$[\mu, \mu]^R$ is not constant

Slowing down sequence $(R^k)_{k\geq 1}$ of reference path measures

Example

R^k : slow Brownian motion on the Riemannian manifold $\mathcal{X}$, $m = \mathrm{vol}$
$L^k = \Delta/(2k)$

Example

R^k : slow simple random walk on the graph $(\mathcal{X}, \sim)$, $m = \sum_{x\in\mathcal{X}} n_x \delta_x$
$L^k u(x) = k^{-1} \sum_{y:y\sim x} [u(y) - u(x)]/n_x$

$L^k = L/k$ is not the only way to slow down

Cooling the heat bath

- The whole particle system performs a rare event to travel from μ_0 to μ_1.
  - Cooperative behavior
  - Gibbs conditioning principle (thermodynamical limit: $N \to \infty$)
  - Solve (S)
- If in addition the temperature tends down to zero, each individual particle faces a harder task. It must travel along an approximate geodesic
  - Individual behavior
  - Large deviation principle (slowing down limit: $k \to \infty$)
  - Find $G^{xy} := \lim_{k \to \infty} R^{k,xy}$

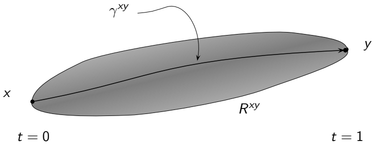
The asymptotic behavior of the slowing down sequence $(R^k)_{k \geq 1}$ encodes geometry

Cooling the heat bath

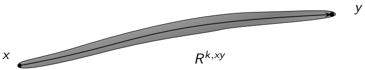
With $L^k = \Delta/(2k)$, $\qquad G^{xy} := \lim_{k\to\infty} R^{k,xy} = \delta_{\gamma^{xy}}$

γ^{xy} : constant speed geodesic

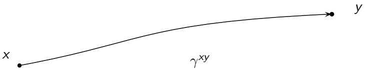
When $k = 1$:



At a lower temperature:



At zero temperature:



Piecewise constant geodesics on a graph

- length $\quad \ell(\omega) := \sum_{0\le t\le 1} \mathbf{1}_{\{\omega_{t^-}\neq\omega_t\}} d(\omega_{t^-},\omega_t)$
- intrinsic distance $\quad d(x,y) = \inf\{\ell(\omega) : \omega\in\Omega, \omega_0 = x, \omega_1 = y\}$

Piecewise constant geodesics

$\Gamma^{xy} := \{\omega\in\Omega; \omega_0 = x, \omega_1 = y, \ell(\omega) = d(x,y)\}$
$\Gamma := \cup_{x,y}\Gamma^{xy}$

Defining a regular geodesic on a discrete metric graph

- non-directed discrete metric graph $(\mathcal{X}, \sim, d)$

Problem

How to define a constant speed geodesic on $(\mathcal{X}, \sim)$?

Strategy

- take a sequence of lazy random walks $(R^k)_{k\geq 1}$
- compute $G^{xy} = \lim_{k\to\infty} R^{k,xy}$
- check that $G^{xy}(\Gamma^{xy}) = 1$
- define $[\delta_x, \delta_y]^{\mathrm{disp}} := \lim_{k\to\infty} [\delta_x, \delta_y]^{R^k} = \lim_{k\to\infty} (R_t^{k,xy})_{0\leq t\leq 1}$

Our aim

- define $[\mu_0, \mu_1]^{\mathrm{disp}}$ as the limit of $[\mu_0, \mu_1]^{R^k}$, $k \to \infty$
- requires to find a relevant sequence $(R^k)_{k\geq 1}$ of lazy random walks

Lazy random walks

We wish to treat $(\mathcal{X}, \sim)$ as a metric measure space $(\mathcal{X}, d, m)$

- to recover d:
  - slow down the walk
  - conditionning at $t = 0$ and $t = 1$
- reference walk: $R \in \mathcal{P}(\Omega)$ with jump kernel $J_x(dy) = \sum_{y:y\sim x} J_x(y)\,\delta_y$

Lazy random walks R^k, $k \to \infty$

$$J_x^k(dy) = \sum_{y:y\sim x} k^{-d(x,y)}\; J_x(y)\,\delta_y$$

Theorem. (Convergence of bridges)

$\lim_{k\to\infty} R^{k,xy} = G^{xy} \in \mathcal{P}(\Gamma^{xy})$ with $G := \mathbf{1}_\Gamma e^{\int_0^1 J_{X_t}(\mathcal{X})\,dt}\, R$

Proof based on large deviations, Girsanov's formula and Γ-convergence

The discrete metric measure graph $(\mathcal{X}, d, m)$

- to be respectful of m, any R^k should be m-stationary
  - take a measure $m = (m_x)_{x\in\mathcal{X}}$.
  - choose $J_x(y) = s(x,y)\sqrt{\frac{m_y}{m_x}}$, s symmetric (many choices)
  - R^k is m-reversible, for any $k \geq 1$

$(\mathcal{X}, d, m)$

The asymptotic behaviour of $(R^k)_{k\geq 1}$ allows for seeing $(\mathcal{X}, d, m)$ as a metric measure space

- interest
  - concentration of the measure m
  - isoperimetry

Slowing down

Dynamical Schrödinger problem

$$H(P|R^k)/\log k \to \min; \quad P \in \mathcal{P}(\Omega) : P_0 = \mu_0, P_1 = \mu_1 \qquad (\mathrm{S}^k_{\mathrm{dyn}})$$

Dynamical Monge-Kantorovich problem

$$\int_\Omega \ell \, dP \to \min; \quad P \in \mathcal{P}(\Omega) : P_0 = \mu_0, P_1 = \mu_1 \qquad (\mathrm{MK}_{\mathrm{dyn}})$$

Theorem

- $\Gamma\text{-}\lim_{k\to\infty}(\mathrm{S}^k_{\mathrm{dyn}}) = (\mathrm{MK}_{\mathrm{dyn}})$
- $\lim_{k\to\infty} \inf(\mathrm{S}^k_{\mathrm{dyn}}) = \inf(\mathrm{MK}_{\mathrm{dyn}})$
- $\lim_{k\to\infty} P^k = P$: singled out solution of $(\mathrm{MK}_{\mathrm{dyn}})$

Slowing down

Schrödinger problem

$$H(\pi|R_{01}^k)/\log k \to \min; \quad \pi \in \mathcal{P}(\mathcal{X}^2) : \pi_0 = \mu_0, \pi_1 = \mu_1 \qquad (S^k)$$

Monge-Kantorovich problem

$$\int_{\mathcal{X}^2} d\, d\pi \to \min; \quad \pi \in \mathcal{P}(\mathcal{X}^2) : \pi_0 = \mu_0, \pi_1 = \mu_1 \qquad (\mathrm{MK}_1)$$

Theorem

- $\Gamma\text{-}\lim_{k\to\infty}(\mathrm{S}^k) = (\mathrm{MK})$
- $\lim_{k\to\infty} \inf(\mathrm{S}^k) = \inf(\mathrm{MK}) := W_1(\mu_0, \mu_1) = \inf(\mathrm{MK}_{\mathrm{dyn}})$
- $\lim_{k\to\infty} \pi^k = \pi$: singled out solution of (MK_1)

Slowing down

Convergence schema

$$\begin{array}{ccccc} P^k(d\omega) & = & \int_{\mathcal{X}^2} & R^k(d\omega \mid X_0 = x, X_1 = y) & \pi^k(dxdy) \\ \downarrow & & & \downarrow & \downarrow \\ P(d\omega) & = & \int_{\mathcal{X}^2} & G^{xy}(d\omega) & \pi(dxdy) \end{array}$$

Entropic interpolations converge to displacement interpolation

$$\begin{array}{ccccc} \mu_t^k(dz) & = & \int_{\mathcal{X}^2} & R_t^k(dz \mid X_0 = x, X_1 = y) & \pi^k(dxdy) \\ \downarrow & & & \downarrow & \downarrow \\ \mu_t(dz) & = & \int_{\mathcal{X}^2} & G_t^{xy}(dz) & \pi(dxdy) \end{array}$$

Definition. Displacement interpolation

$$[\mu_0, \mu_1]^{\mathrm{disp}} := (\mu_t)_{0 \le t \le 1}$$

Interpolations

- On the discrete metric graph $(\mathcal{X}, \sim, d)$
  - $R^k \to J^k_x(dy) = \sum_{y:y\sim x} k^{-d(x,y)} J_x(y)\,\delta_y$
  - $G^{xy} = \lim_{k\to\infty} R^{k,xy}$
  - $\pi = \lim_{k\to\infty} \pi_k$ singled out solution of $(\mathrm{MK})_1$

Displacement interpolation on a graph

$$\mu_t(\cdot) = \int_{\mathcal{X}^2} G^{xy}_t(\cdot)\pi(dxdy)$$

Reference: L. , arXiv:1308.0226

- On the Riemannian manifold $(\mathcal{X}, g)$
  - $R^k \to L^k = \Delta/(2k)$
  - $\delta_{\gamma^{xy}} = \lim_{k\to\infty} R^{k,xy}$
  - $\pi = \lim_{k\to\infty} \pi_k$ singled out solution of $(\mathrm{MK})_2$

Displacement interpolation on a manifold

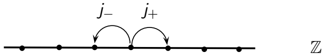
$$\mu_t(\cdot) = \int_{\mathcal{X}^2} \delta_{\gamma^{xy}_t}(\cdot)\pi(dxdy)$$

References: Mikami, L. (2012)

Interpolations

As an example, take

- $\mathcal{X} = \mathbb{Z}$
- $J_x = j_- \delta_{(x-1)} + j_+ \delta_{(x+1)}, \quad j_-, j_+ > 0$:

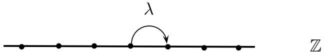


- $L^k = L/k$: uniform slowing down

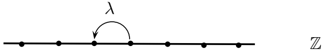
Then,

- $m_x = (j_+/j_-)^x, \quad x \in \mathbb{Z}$
- $d(x,y) = |y - x|, \quad x, y \in \mathbb{Z}$
- if $x \le y$: $G^{xy} = \overrightarrow{R}^{xy}$; if $x \ge y$: $G^{xy} = \overleftarrow{R}^{xy}$ where

$\overrightarrow{R}$ is increasing Poisson:



$\overleftarrow{R}$ is decreasing Poisson:



Benamou-Brenier formula (displacement interpolation)

Theorem

- $W_1(\mu_0, \mu_1) = \inf_{(\nu,K)} \int_0^1 dt \sum_{x,y:x\sim y} d(x,y) K_{t,x}(y)\, \nu_t(x)$

  (ν, K) : $\quad \partial_t \nu_t(x) - \sum_{y:y\sim x} \{\nu_t(y) K_{t,y}(x) - \nu_t(x) K_{t,x}(y)\} = 0$

  $\qquad \nu_0 = \mu_0, \nu_1 = \mu_1$
- inf attained at the displacement interpolation:

  $\nu = \mu,\ K = J^\mu$

$$W_1(\mu_0, \mu_1) = \int_0^1 dt \sum_{x,y:x\sim y} d(x,y) J^{\mu}_{t,x}(y)\, \mu_t(x)$$

Constant speed interpolations

$$W_1(\mu_0, \mu_1) = \int_{[0,1]} \mathrm{speed}(\mu)_t \, dt$$

- $\mathrm{speed}(\mu)_t = \sum_{x,y:x\sim y} d(x,y) J^{\mu}_{t,x}(y) \, \mu_t(x)$
- $\tau : [0,1] \to [0,1]$: change of time
- $J^{\mu\circ\tau}_t = \tau'(t) J^{\mu}_{\tau(t)}$
- $W_1(\mu_0, \mu_1) = \int_{[0,1]} \mathrm{speed}(\mu \circ \tau)_t \, dt, \quad \forall \tau$

Proposition

There exists a unique change of time τ such that

$$\mathrm{speed}(\mu \circ \tau)_t = W_1(\mu_0, \mu_1), \quad \forall t$$

τ depends on (μ_0, μ_1)

A conserved quantity

- $\nu K_t(\mathcal{X}^2) := \sum_{x,y:y\sim x} \nu_t(x) K_{t,x}(y)$

Average rate of mass displacement

$$\mu J_t^\mu(\mathcal{X}^2) := \sum_{x,y:y\sim x} \mu_t(x) J_{t,x}^\mu(y)$$

Proposition (entropic interpolation)

$$\mu J_t^\mu(\mathcal{X}^2) - \mu J_t(\mathcal{X}^2) = constant, \ \forall t$$

Theorem (displacement interpolation)

$$\mu J_t^\mu(\mathcal{X}^2) := \sum_{x,y:y\sim x} \mu_t(x) J_{t,x}^\mu(y) = constant, \ \forall t$$

Thank you for your attention