Monotonicity Formulas for Bakry-Emery Ricci Curvature

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Smooth Metric Measure Spaces

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By O'Neill's formula, the Ricci curvature of the warped product metric g_ϵ in the M direction is

$$\mathrm{Ric}_M + \mathrm{Hess} f - \frac{1}{m} df \otimes df.$$

m-Bakry-Emery Ricci tensor

Therefore for smooth metric measure spaces $(M^n, g, e^{-f} dvol_g)$, the corresponding Ricci tensor is

$$\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess} f - \frac{1}{m} df \otimes df \quad \text{for } 0 \leq m \leq \infty,$$

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If $m_1 \geq m_2$, then $\mathrm{Ric}_f^{m_1} \geq \mathrm{Ric}_f^{m_2}$.

So $\mathrm{Ric}_f^m \geq \lambda g$ implies $\mathrm{Ric}_f \geq \lambda g$.

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When m is finite

$$\frac{1}{2}\Delta_f |\nabla u|^2 \geq \frac{(\Delta_f(u))^2}{m+n} + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f^m(\nabla u, \nabla u).$$

Theorem (Qian1997, Laplacian comparison)

If $Ric_f^m \geq (n+m-1)H$, *then* $\Delta_f(r) \leq \Delta_H^{n+m}(r)$.

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Theorem (Qian1995, Gradient Estimate)

If $Ric_f^m \geq 0$, and u is a positive function with $\Delta_f u = 0$ on $B(x, R)$, then on $B(x, \frac{R}{2})$

$$|\nabla \log u| \leq \frac{C(n+m)}{R}.$$

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Theorem (Wei-Wylie2009, JDG)

If $Ric_f \geq (n-1)H$ and $|f| \leq k$ then

$$\Delta_f(r) \leq \Delta_H^{n+4k}(r) \quad (\text{for } H > 0 \text{ assume } r \leq \pi/4\sqrt{H})$$

Green's Function

Definition

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G always exists. (Malgrange1955, Li-Tam1987)

After normalization, near the pole, for $n \geq 3$,

$$G(y) = d^{2-n}(x_0, y)(1 + o(1)),$$

$$|\nabla G(y)| = (n-2)d^{1-n}(x_0, y)(1 + o(1)).$$

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When $f = 0$ and $\mathrm{Ric} \geq 0$, (Varopoulos 1981)

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Proposition (Song-W-Wu)

If a smooth metric measure space $(M^n, g, e^{-f} dvol)$ $(n \geq 3)$ has $Ric_f^m \geq 0$, then for $k = n + m$, $\exists r_0 > 0$, such that

$$|\nabla b(y)| \leq C(n, m, r_0) \quad \text{on } M \setminus B(x_0, r_0).$$

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- This can not be true when $k > n$ as $|\nabla b(y)| \to \infty$ as $y \to x_0$.
- For $\mathrm{Ric}_f \geq 0$, $|\nabla b(y)|$ may not be bounded as $y \to \infty$ (for any k).

The f-Laplacian of b and $|\nabla b|$

Proposition

$\forall q, \beta \in \mathbb{R}$,

$$\begin{aligned}\Delta_f(b^{2q}|\nabla b|^\beta) &= \frac{\beta}{4} b^{2q-2}|\nabla b|^{\beta-2}\Big\{ \left|Hess\, b^2\right|^2 + Ric_f(\nabla b^2, \nabla b^2) \\ &+2(k-2+2q)\langle \nabla b^2, \nabla |\nabla b|^2\rangle + 4(\beta-2) b^2 |\nabla|\nabla b||^2 \\ &+ \left[\frac{8q}{\beta}(k-2+2q) - 4k\right]|\nabla b|^4 \Big\}\end{aligned}$$

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(holds for any positive f-harmonic function G, not just Green's function.)

Ratio of Area and Volume: A_f, V_f

For any $l, \beta, p \in \mathbb{R}$, $r > 0$, consider

$$A_f^\beta(r) = r^{1-l} \int_{b=r} |\nabla b|^{\beta+1} e^{-f},$$

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When $k = l = n, p = 2$, these are A_β, V_β in Colding-Minicozzi.

Derivatives of A_f, V_f

Lemma

$$
\begin{aligned}
(V_f^{\beta,p})'(r) &= \frac{p-l}{r} V_f^{\beta,p}(r) + \frac{1}{r} A_f^{\beta}(r), \\
(A_f^{\beta})'(r) &= \frac{k-l-2+p}{r} A_f^{\beta}(r) \\
&\quad + r^{p-1-l} \int_{b \leq r} \left(\Delta_f (b^{2-p} |\nabla b|^{\beta}) \right) e^{-f}.
\end{aligned}
$$

First Monotonicity Formula

Theorem

When $k > 2$, $C(n,k,p) > 0$, for any $\alpha \in \mathbb{R}$,

$$(A_f^\beta - \alpha V_f^{\beta,p})'(r) = r^{p-1-l} \int_{b \leq r} \frac{\beta |\nabla b|^{\beta-2}}{4 b^p} \Big\{ \left| Hess\, b^2 \right|^2 + Ric_f(\nabla b^2, \nabla b^2) + 4(\beta-2) b^2 |\nabla |\nabla b||^2 \Big\} e^{-f} + \frac{1}{r}\left(\lambda_1 A_f^\beta(r) + \lambda_2 V_f^{\beta,p}(r) \right),$$

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where

$$\lambda_1 = 3k - p - l - 2 - \alpha,$$
$$\lambda_2 = (p + 2 - 2k)(k - p) - \beta k - \alpha(p - l).$$

Second Monotonicity Formula

Theorem

For $c, d \in \mathbb{R}$, let $g(r) = r^c \left(r^d A_f^{\beta}(r)\right)'$. Then for $0 < r_1 < r_2$,

$$\begin{aligned} g(r_2) - g(r_1) &= \int_{r_1 \leq b \leq r_2} \frac{\beta}{4} b^{c+d-l-1} |\nabla b|^{\beta-2} \Big\{ |Hess\, b^2|^2 + Ric_f(\nabla b^2, \nabla b^2) \\ &\quad + \lambda_3 |\nabla b|^4 + 4(\beta - 2) b^2 |\nabla |\nabla b||^2 \Big\} e^{-f} \\ &\quad + \lambda_4 \int_{r_1 \leq b \leq r_2} b^{c+d-l} \langle \nabla b, \nabla |\nabla b|^{\beta} \rangle e^{-f}, \end{aligned}$$

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\end{aligned}
$$

$$
\begin{aligned}
\lambda_3 &= \frac{4}{\beta}(k+d-l+c-1)(k+d-l) - 4k, \\
\lambda_4 &= 3k - 2l - 3 + c + 2d.
\end{aligned}
$$

Monotonicity for $\mathrm{Ric}_f^m \geq 0$

Since

$$\mathrm{Ric}_f(\nabla b^2, \nabla b^2) = \mathrm{Ric}_f^m(\nabla b^2, \nabla b^2) + \frac{\langle \nabla b^2, \nabla f \rangle^2}{m}$$

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we have

$$\begin{aligned} \left|\mathrm{Hess}\, b^2\right|^2 + \mathrm{Ric}_f(\nabla b^2, \nabla b^2) &\geq \frac{(\Delta b^2)^2}{n} + \frac{\langle \nabla b^2, \nabla f \rangle^2}{m} \\ &\geq \frac{\left(\Delta_f b^2\right)^2}{n+m} = \frac{4k^2}{n+m} |\nabla b|^4. \end{aligned}$$

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If $M^n (n \geq 3)$ has $Ric_f^m \geq 0$, then, for $k \geq n+m$, $\beta \geq 2$, $k \leq l \leq 2k-2$, $\alpha = 3k-p-l-2$ and $C(n,k,p) > 0$,

$$(A_f^\beta - \alpha V_f^{\beta,p})'(r)$$

$$\geq r^{p-1-l} \int_{b \leq r} \frac{\beta |\nabla b|^{\beta-2}}{4 b^p} \left\{ \left| Hess\, b^2 - \frac{\Delta b^2}{n} g \right|^2 \right\} e^{-f} \geq 0.$$

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When $m = 0$ (i.e. f is constant and $\mathrm{Ric} \geq 0$), and $p = 0, \beta = 2$, $k = l = n$, this is the first monotonicity formula in Colding2012.

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When $m = 0$ (i.e. f is constant and $\mathrm{Ric} \geq 0$), and $p = 0, \beta = 2$, $k = l = n$, this is the first monotonicity formula in Colding2012.

L.-F. Wang2013 proved this when $p = 0, \beta = 2$, $k = l = n + m$.

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If $M^n (n \geq 3)$ has $\text{Ric}_f^m \geq 0$, then for $\beta \geq 2$, $k = l = n + m$, $(A_f^\beta)'(r) \leq 0$ and $(V_f^{\beta,p})'(r) \leq 0$ for $p < n + m - \frac{\beta m}{n-2}$.

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In fact

$$(A_f^\beta)'(r) \leq -\frac{\beta}{4} r^{k-3} \int_{b \geq r} b^{2-2k} |\nabla b|^{\beta-2} \left| Hess\, b^2 - \frac{\Delta b^2}{n} g \right|^2 e^{-f}.$$

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Again this reduces to a formula in Colding when $k = l = n$ and f is constant.

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Theorem

If $M^n (n \geq 4)$ has $Ric_f \geq 0$, then

for $\beta = 2, p = 0, k \geq 12, l = \frac{3}{2}k - 1$, we have
$(A_f^\beta - (\frac{3}{2}k - 1)V_f^{\beta,p})'(r) \geq 0$;

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for $\beta = 2, k \geq 12, l = \frac{3}{2}(k-1), r_2 \geq r_1 > 0$, we have
$(A_f^\beta)'(r_2) \geq (A_f^\beta)'(r_1)$.

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for $\beta = 2, p = 0, k \geq 12, l = \frac{3}{2}k - 1$, we have
$(A_f^\beta - (\frac{3}{2}k - 1)V_f^{\beta,p})'(r) \geq 0$;

for $\beta = 2, k \geq 12, l = \frac{3}{2}(k-1)$, $r_2 \geq r_1 > 0$, we have
$(A_f^\beta)'(r_2) \geq (A_f^\beta)'(r_1)$.

Here $l > k$.

When $k = l = n$

Theorem

When $k = l = n, \beta = 2$, for $0 < r_1 < r_2$,

$$r_2^{3-n}(A_f^\beta)'(r_2) - r_1^{3-n}(A_f^\beta)'(r_1)$$
$$= \int_{r_1 \leq b \leq r_2} \frac{1}{2} b^{2-2n} \left\{ \left| Hess\, b^2 - \frac{\Delta b^2}{n} g \right|^2 + Ric_f(\nabla b^2, \nabla b^2) \right.$$
$$\left. + 4|\nabla b|^2 \langle \nabla b^2, \nabla f \rangle + \frac{\langle \nabla b^2, \nabla f \rangle^2}{n} \right\} e^{-f}.$$

Bryant Soliton

Example

For the Bryant soliton, when $k = l = n, \beta = 2$, *for* $r_2 \geq r_1 >> 0$,

$$\begin{aligned} r_2^{3-n}(A_f^{\beta})'(r_2) &\geq r_1^{3-n}(A_f^{\beta})'(r_1) \text{ if } n = 3, \\ r_2^{3-n}(A_f^{\beta})'(r_2) &\leq r_1^{3-n}(A_f^{\beta})'(r_1) \text{ if } n \geq 5. \end{aligned}$$

For a positive function f on M with $\int_M f = 1$, Shannon entropy

$$S_0 = -\int_M f \log f,$$

Fisher information

$$F_0 = \int_M \frac{|\nabla f|^2}{f}.$$

If $u > 0$ is a solution of the heat equation

$$(\partial_t - \Delta) u = 0$$

with $\int_M u = 1$, then

$$S(t) = S_0(t) - \frac{n}{2}\log t = -\int_M u \log u - \frac{n}{2}\log t$$

$$F(t) = tF_0(t) - \frac{n}{2}.$$

Let $W = S + F$, then W and S are nonincreasing when $\mathrm{Ric} \geq 0$.