Spaces with Ricci curvature bounded from below

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August 30, 2013

Lessons

1) On the definition of spaces with Ricci curvature bounded from below

2) Analytic properties of $RCD(K, N)$ spaces

3) Geometric properties of $RCD(K, N)$ spaces

Geometric properties of $RCD(K, N)$ spaces

- The Abresch-Gromoll inequality
- The splitting theorem
  - Statement
  - The proof in the smooth case
  - The proof in the non-smooth case
- The maximal diameter theorem
- Tangent spaces as mGH limits of blow-ups

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The Abresch-Gromoll inequality

On a Riemannian manifold with $\mathrm{Ric} \geq K$ and $\dim \leq N$ we have

$$E(x) \leq f_{K,N}(h(x)), \qquad \text{provided} \qquad h(x) \leq \frac{\min\{\mathsf{d}(x,\gamma_0),\mathsf{d}(x,\gamma_1)\}}{2}$$

for some (explicitly given) $f_{K,N}$ satisfying

$$\lim_{h \downarrow 0} \frac{f_{K,N}(h)}{h} = 0.$$

Ingredients of the proof

Laplacian comparison estimates for the distance

Linearity of the Laplacian

Weak maximum principle

The non-smooth case

Repeating verbatim the proof on $RCD(K,N)$ spaces we obtain:

Thm. (G., Mosconi '12) The Abresch-Gromoll inequality holds in the non-smooth setting in the same form as in the smooth one.

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The splitting theorem

Thm. (Cheeger-Gromoll '71)
Let M be a Riemannian manifold with $\mathrm{Ric} \geq 0$ which contains a line.
Then $M = N \times \mathbb{R}$ for some Riemannian manifold N.

The almost splitting

Thm. (Cheeger-Colding '96) Let M be a Riemannian manifold with $\mathrm{Ric} \geq -\varepsilon$ which contains a geodesic with length L, with $\varepsilon, L^{-1} \ll 1$

Then 'a big portion of M is mGH-close to a product'

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Then 'a big portion of M is mGH-close to a product'

Cor. (Splitting for limit spaces) Let $(X, \mathsf{d}, \mathfrak{m})$ be a pmGH of Riemannian manifolds (M_n) with uniformly bounded dimension and with $\mathrm{Ric}(M_n) \geq -\varepsilon_n$, where $\varepsilon_n \downarrow 0$.

Assume that X contains a line. Then it splits off a factor $\mathbb{R}$

The non-smooth splitting

Thm. (G. '13) Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD(0, N)$ space containing a line. Then there is a space $(X', \mathsf{d}', \mathfrak{m}')$ such that

$$(X, \mathsf{d}, \mathfrak{m}) \quad \text{is isomorphic to} \quad (X' \times \mathbb{R}, \mathsf{d}' \otimes \mathsf{d}_{\mathrm{Eucl}}, \mathfrak{m}' \times \mathcal{L}^1)$$

where

$$(\mathsf{d}' \otimes \mathsf{d}_{\mathrm{Eucl}})\big((x', t), (y', s)\big) := \sqrt{\mathsf{d}'(x', y')^2 + |t - s|^2}$$

Moreover:

- If $N \geq 2$ then $(X', \mathsf{d}', \mathfrak{m}')$ is an $RCD(0, N-1)$ space
- If $N \in [1, 2)$ then X' contains only one point

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The Busemann function

Let $\gamma : [0, \infty) \to M$ an half line.
The Busemann function b associated to it is

$$\mathrm{b}(x) := \lim_{t \to +\infty} t - \mathrm{d}(x, \gamma_t) = \sup_{t \geq 0} t - \mathrm{d}(x, \gamma_t)$$

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If $\gamma : (-\infty, +\infty) \to M$ is a line we can associate to it 2 Busemann functions

$$\mathrm{b}^+(x) := \lim_{t \to +\infty} t - \mathrm{d}(x, \gamma_t)$$
$$\mathrm{b}^-(x) := \lim_{t \to +\infty} t - \mathrm{d}(x, \gamma_{-t})$$

Effect of $\mathrm{Ric} \geq 0$ on the Busemann function for an half line

If $\mathrm{Ric} \geq 0$ and $\bar{x} \in M$

$$\Delta \frac{\mathsf{d}^2(\cdot, \bar{x})}{2} \leq \dim(M)$$

Hence

$$\Delta \mathsf{d}(\cdot, \gamma_t) \leq \frac{\dim(M)}{\mathsf{d}(\cdot, \gamma_t)}$$

Passing to the limit we obtain

$$\Delta \mathrm{b} \geq 0,$$

i.e. the b is subharmonic.

What for the Busemann function for a line

b^+ and b^- are subharmonic, thus so is $\mathrm{b}^+ + \mathrm{b}^-$.
The triangle inequality gives

$$\mathrm{b}^+ + \mathrm{b}^- \leq 0$$

and the fact that γ is a line ensures that

$$(\mathrm{b}^+ + \mathrm{b}^-)(\gamma_0) = 0$$

hence (strong maximum principle) it holds

$$\mathrm{b}^+ + \mathrm{b}^- \equiv 0$$

and in particular b^+ and b^- are harmonic

Use of the Bochner equality and inequality

For any f smooth it holds

$$\Delta \frac{|\nabla f|^2}{2} = \|\mathrm{Hess}\, f\|^2_{\mathrm{HS}} + \nabla f \cdot \nabla \Delta f + \mathrm{Ric}(\nabla f, \nabla f)$$
$$\geq \frac{(\Delta f)^2}{\dim(M)} + \nabla f \cdot \nabla \Delta f$$

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For b^+ we have $|\nabla \mathrm{b}^+| \equiv 1$ and $\Delta \mathrm{b}^+ \equiv 0$ and thus the equality

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$$\Delta \frac{|\nabla \mathrm{b}^+|^2}{2} = \frac{(\Delta \mathrm{b}^+)^2}{\dim(M)} + \nabla \mathrm{b}^+ \cdot \nabla \Delta \mathrm{b}^+$$

which yields

$$\|\mathrm{Hess}\, \mathrm{b}^+\|_{\mathrm{HS}}^2 \equiv \frac{(\Delta \mathrm{b}^+)^2}{\dim(M)} \equiv 0$$

i.e. b^+ is both convex and concave.

Isometries via gradient flows

Since b^+ is convex, its gradient flow contracts distances.

Since $\mathrm{b}^+ = -\mathrm{b}^-$ is concave, its gradient flow expands distances.

Thus the gradient flow of b^+ produces a 1-parameter family of isometries.

Conclusion of the argument

Put $N := \{\mathrm{b}^+ = 0\}$

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For $x \in N$ and $v \in T_x N$ it is obvious that $v \cdot \nabla \mathrm{b}^+(x) = 0$ and the conclusion follows from the fact the gradient flow of b^+ is a 1-parameter family of isometries.

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Harmonicity of $\mathrm{b}^{\pm}$

As in the smooth case, from the Laplacian comparison estimate we deduce

$$\Delta \mathrm{b}^{\pm} \geq 0,$$

and using the strong maximum principle (Bjorn-Bjorn '07) we obtain

$$\mathrm{b}^{+} + \mathrm{b}^{-} = 0$$

i.e.

$$\Delta \mathrm{b}^{\pm} = 0$$

Gradient flow of $\mathrm{b}^{\pm}$ and geodesics

For every $t \in \mathbb{R}$ the function $t\mathrm{b}^{+}$ is c-concave and

$$(t\mathrm{b}^{+})^{c} = t\mathrm{b}^{-} - \frac{t^2}{2}$$

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Hence from existence and uniqueness of optimal maps we deduce that

$$\text{for } \mathfrak{m}\text{-a.e. } x \in X \text{ there is a unique } F_t(x) \in \partial^c(t\mathrm{b}^{+})(x)$$

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and we can check that for $\mathfrak{m}$-a.e. x we have

$$\begin{aligned} t \quad \mapsto \quad F_t(x), & \qquad \text{is a line} \\ F_{t+s}(x) = F_t(F_s(x)), & \qquad \forall t, s \in \mathbb{R} \end{aligned}$$

Measure preservation

For every $\mu = \rho\mathfrak{m} \ll \mathfrak{m}$ the map $[0,1] \ni t \mapsto (F_t)_\sharp\mu$ is a W_2-geodesic induced by b^+.

Arguing as in the proof of the Laplacian comparison estimates we deduce

$$\frac{1}{t}\big(\mathcal{U}_N((F_t)_\sharp\mu) - \mathcal{U}_N(\mu)\big) \geq -\frac{1}{N}\int \nabla(\rho^{1-\frac{1}{N}}) \cdot \nabla \mathrm{b}^+ \, \mathrm{d}\mathfrak{m} = 0$$

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Switching b^+ and b^- we deduce

$$\mathcal{U}_N((F_t)_\sharp\mu) = \mathcal{U}_N(\mu), \qquad \forall t \in \mathbb{R}$$

and thus

$$(F_t)_\sharp\mathfrak{m} = \mathfrak{m}, \qquad \forall t \in \mathbb{R}$$

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Write it with $\mathrm{b}^+ + \varepsilon f$ in place of f we obtain

$$\Delta(\nabla \mathrm{b}^+ \cdot \nabla f) = \nabla \mathrm{b}^+ \cdot \nabla \Delta f$$

for every f ‘smooth enough’.

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for every f 'smooth enough'.
Using this identity in computing $\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int |D(f \circ F_t)|^2 \,\mathrm{d}\mathfrak{m}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int |D(f \circ F_t)|^2 \,\mathrm{d}\mathfrak{m} = 0$$

and thus

$$t \mapsto \frac{1}{2}\int |D(f \circ F_t)|^2 \,\mathrm{d}\mathfrak{m} \qquad \text{is constant}$$

Isomorphisms by duality

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Let $(X_1, \mathrm{d}_1, \mathfrak{m}_1)$, $(X_2, \mathrm{d}_2, \mathfrak{m}_2)$ be $RCD(K, \infty)$ spaces and $T : X_1 \to X_2$ Borel and a.e. invertible.

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Then T is (up to a modification on a negligible set) an isomorphism if and only if

$$\|f \circ T\|_{W^{1,2}(X_1)} = \|f\|_{W^{1,2}(X_2)}, \qquad \forall f : X_2 \to \mathbb{R}$$

The quotient space

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We can then declare $x \sim y$ if $x = F_t(y)$ for some $t \in \mathbb{R}$, put $X' := X/\sim$ and define

$$\mathrm{d}'\big(\pi(x), \pi(y)\big) := \inf_{t \in \mathbb{R}} \mathrm{d}(x, F_t(y)) \qquad \forall x, y \in X$$

and

$$\mathfrak{m}'(E) := \mathfrak{m}(\pi^{-1}(E)) \cap \mathrm{b}^{-1}([0,1]) \qquad \forall E \subset X' \text{ Borel}$$

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Define $\iota : X' \to X$ as

$$\iota(x') = x \qquad \text{if} \qquad \pi(x) = x' \quad \text{and} \quad \mathrm{b}^{+}(x) = 0.$$

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Problem: is ι an isometry?

How to gain C^1 regularity

Let (μ_t) be a geodesic such that $\mu_t \leq C\mathfrak{m}$ for every $t \in [0, 1]$ and φ_t s.t. $-(1-t)\varphi_t$ is a Kantorovich potential from μ_t to μ_1

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Then:
For $f \in W^{1,2}(X)$ the map $t \mapsto \int f \, \mathrm{d}\mu_t$ is C^1 and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \, \mathrm{d}\mu_t = \int \nabla f \cdot \nabla \varphi_t \, \mathrm{d}\mu_t, \qquad \forall t \in [0,1]$$

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For $\nu \in \mathscr{P}_2(X)$ the map $t \mapsto \frac{1}{2}W_2^2(\mu_t,\nu)$ is C^1 and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu) = \int \nabla\phi_t \cdot \nabla\varphi_t\,\mathrm{d}\mu_t, \qquad \forall t \in [0,1]$$

where ϕ_t is a Kantorovich potential from μ_t to ν.

Basic properties of $(X', \mathrm{d}', \mathfrak{m}')$

Arguing as in the smooth case but at the level of probability measures $\mu, \nu \leq C\mathfrak{m}$ we deduce that

$$\text{the minimum of } t \mapsto \frac{1}{2} W_2^2((F_t)_\sharp \mu, \nu)$$
$$\text{is attained at that } t_0 \text{ such that } \int \mathrm{b}^+ \, \mathrm{d}(F_{t_0})_\sharp \mu = \int \mathrm{b}^+ \, \mathrm{d}\nu$$

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Picking μ, ν with support going to a point we conclude that ι is indeed an isometric embedding.

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Picking μ, ν with support going to a point we conclude that ι is indeed an isometric embedding.

It is then easy to see that $(X', \mathsf{d}', \mathfrak{m}')$ is an $RCD(0, N)$ space

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The first follows using again the duality with Sobolev functions

The second by a general dimension-reduction argument introduced by Cavalletti-Sturm

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The statement

Thm. (Ketterer - announced) Let $(X, \mathrm{d}, \mathfrak{m})$ be an $RCD^*(N-1, N)$ space containing two points at distance π.
Then it is isomorphic to a spherical suspension over a space $(X', \mathrm{d}', \mathfrak{m}')$. Moreover:

- If $N \geq 2$ then $(X', \mathrm{d}', \mathfrak{m}')$ is an $RCD^*(N-2, N-1)$ space
- If $N \in [1, 2)$ then X' contains either only one point or two points at distance π.

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Crucial ingredient of the proof:
TFAE

- $(X, \mathsf{d}, \mathfrak{m})$ is an $RCD^*(N-1, N)$ space
- The metric cone built over $(X, \mathsf{d}, \mathfrak{m})$ is an $RCD(0, N+1)$ space

Proved via the study of the Bochner inequality

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Tangent spaces

What we expect:
Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD(K, N)$ space. Then for $\mathfrak{m}$-a.e. point x the rescaled space pointed at x converge to $\mathbb{R}^n$, for some $n \leq N$ independent on x.

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What we have:
Thm. (G., Mondino, Rajala '13)
Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD(K, N)$ space. Then for $\mathfrak{m}$-a.e. point x there exists a sequence of rescaling such that the rescaled space pointed at x converge to $\mathbb{R}^n$, for some $n \leq N$ possibly dependent on x and the chosen sequence of scalings.

Idea of the proof

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(5) Use Preiss' principle:

> *'Iterated tangents are tangents'*

Thank you