Lagrangian Solutions for Semigeostrophic System with Singular Initial Data

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Derivation of Semigeostrophic Model

Boussinesq system with Coriolis force in 3D:

$$
\begin{aligned}
&\frac{D}{Dt}(u_1, u_2) + (\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}) = (u_2, \, -u_1), \\
&\frac{D}{Dt}\rho = 0, \\
&\text{div} u = 0, \\
&\frac{\partial p}{\partial x_3} + \rho = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
&\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla, \\
&u = (u_1, u_2, u_3), \qquad \nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})
\end{aligned}
$$

Derivation of Semigeostrophic Model

Strong Coriolis forcing:

$$\begin{aligned}
&\varepsilon \frac{D}{Dt}(u_1, u_2) + (\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}) = (u_2, -u_1), \\
&\frac{D}{Dt}\rho = 0, \\
&\text{div} u = 0, \\
&\frac{\partial p}{\partial x_3} + \rho = 0,
\end{aligned}$$

where

$$\begin{aligned}
&\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla, \\
&u = (u_1, u_2, u_3), \qquad \nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})
\end{aligned}$$

Derivation of Semigeostrophic Model

Rotation dominated motion: set $\varepsilon = \mathbf{0}$, thus drop inertial term $\frac{D}{Dt}(u_1, u_2)$ to obtain Geostrophic Balance. This defines Geostrophic Velocities (horizontal)

$$(v_1^g,\ v_2^g) = (\frac{\partial p}{\partial x_2}, -\frac{\partial p}{\partial x_1}),$$

Substitute geostrophic velocities into inertial term $\frac{D}{Dt}(v_1^g,\ v_2^g)$ (where still $\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla$), get Semigeostrophic System

Semigeostrophic System in 3D

Model with rigid boundaries i.e. in domain $\Omega \subset \mathbb{R}^3$.

$$
\begin{aligned}
&\frac{D}{Dt}(v_1^g,\, v_2^g) + (\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}) = (u_2,\, -u_1),\\
&\frac{D}{Dt}\rho = 0,\\
&\mathrm{div} u = 0,\\
&\frac{\partial p}{\partial x_3} + \rho = 0,\\
&\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla, \quad (v_1^g,\, v_2^g) = (\frac{\partial p}{\partial x_2}, -\frac{\partial p}{\partial x_1}),
\end{aligned}
$$

in $(0, T) \times \Omega$, with initial and boundary conditions:

$$
\begin{aligned}
&u \cdot \nu = 0 \ \text{ on } \ (0, T) \times \partial\Omega,\\
&p(0, x) = p_0(x) \ \text{ in } \ \{t = 0\} \times \Omega.
\end{aligned}
$$

Semigeostrophic System in 3D

Model was introduced by Eliassen(1948), Hoskins(1975).
Rewrite system:
Use $(u_1, u_2, u_3) = \frac{D}{Dt}(x_1, x_2, x_3)$.
Introduce

$$P(t,x) := p(t,x) + \frac{1}{2}(x_1^2 + x_2^2),$$
$$P_0(x) = p_0(x) + \frac{1}{2}(x_1^2 + x_2^2),$$

and $\frac{\pi}{2}$-rotation in horizontal plane matrix:

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Semigeostrophic System in 3D

SG system takes form:

$$\frac{DX}{Dt} = J(X - x)$$
$$\text{div} u = 0,$$
$$X = \nabla P, \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla,$$

in $(0, T) \times \Omega$, with initial and boundary conditions:

$$u \cdot \nu = 0 \text{ on } (0, T) \times \partial\Omega,$$
$$P(0, x) = P_0(x) \text{ in } \{t = 0\} \times \Omega.$$

Cullen-Purser stability condition: $P(t, \cdot)$ is convex

Dual Space

Dual space: change of variables:

$$(t, x) \to (t, X), \qquad \text{where} \;\; X = \nabla P_t(x),$$

where we use notation $P_t(\cdot) = P(t, \cdot)$.

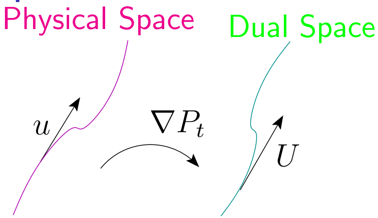
If $P_t(\cdot)$ is strictly convex, then inverse transform is given by

$$x = \nabla P_t^*(X),$$

where $P_t^*(\cdot)$ is the convex dual (Legendre transform) of $P_t(\cdot)$:

$$P_t^*(X) = \sup_{x \in \Omega} [x \cdot X - P(x, t)].$$

Velocity in Dual Space



Let $x(t)$ be a particle path in physical space: $\dot{x}(t) = u(t, x(t))$.
Then $X(t) = \nabla P_t(x(t))$ is a particle path in physical space.
Then velocity in dual space is (using SG system):

$$
\begin{aligned}
U(t, X(t)) = \dot{X}(t) &= \frac{d}{dt}\left(\nabla P_t(x(t))\right) \\
&= \frac{\partial}{\partial t}\nabla P_t(x) + (\dot{x}(t)\cdot\nabla)\nabla P_t(x) \\
&= \frac{\partial}{\partial t}X + (u\cdot\nabla)X = J(X - x) = J(X - \nabla P_t^*(X)).
\end{aligned}
$$

Density in Dual Space

Recall: if μ, ν are measures on metric spaces X, Y, and $r : X \to Y$ is Borel, then r pushes forward μ to ν, denoted $r_{\#}\mu = \nu$ if

$$\mu(r^{-1}(A)) = \nu(A) \quad \text{for each Borel} \;\; A \subset Y.$$

For $t \geq 0$, denote $\alpha_t = \nabla P_{t\#}\chi_\Omega$. Then α_t is density in dual space.

Equation for $\alpha(t, X) = \alpha_t(X)$ (heuristic argument):

Equation $\operatorname{div} u = 0$ in Ω together with condition $u \cdot \nu = 0$ on $\partial\Omega$ imply

$$\partial_t \chi_\Omega + \operatorname{div}(u\chi_\Omega) = 0 \quad \text{in} \;\; \mathbb{R}^3.$$

Then changing variables $X = \nabla P_t(x)$, and using that velocity in dual space is $U(t, X) = J(X - x) = J(X - \nabla P_t^*(X))$, yields

$$\partial_t \alpha + \operatorname{div}(U\alpha) = 0 \quad \text{in} \;\; (0, \infty) \times \mathbb{R}^3.$$

Semigeostrophic system in Dual Space

$$
\begin{aligned}
&\partial_t \alpha + \mathsf{div}(U\alpha) = 0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^3,\\
&U(t,X) = J(X - \nabla P_t^*(X)),\\
&\nabla P_{t\#}\chi_\Omega = \alpha_t,\\
&\alpha_{|t=0} = \alpha_0.
\end{aligned}
$$

Existence in dual space:

Benamou, Brenier (1998) for 3D rigid boundaries model, case $\alpha_0 \in L^q, \quad q \geq 3$.

Cullen, Gangbo (2001) for 2D shallow water SG model, case $\alpha_0 \in L^q, \quad q \geq 1$.

Lopes-Filho, Nussenzveig-Lopes (2002) extended to $q = 1$.

Loeper (2006) case α_0 a measure. Then ∇P^* is replaced by barycentric projection

Ambrosio, Gangbo (2008) case α_0 a measure: SG in dual space is a Hamiltonian ODE in the Wasserstein spaces.

Relation to Monge-Kantorovich mass transport

Semigeostrophic system in Dual Space

$$
\begin{aligned}
&\partial_t \alpha + \mathrm{div}(U\alpha) = 0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^3,\\
&U(t,X) = J(X - \nabla P_t^*(X)),\\
&\nabla P_{t\#}\chi_\Omega = \alpha_t,\\
&\alpha_{|t=0} = \alpha_0.
\end{aligned}
$$

Recall: $P_t(\cdot)$ is convex. Thus $\nabla P_t(\cdot)$ is the optimal map for Monge's problem between measures χ_Ω and α_t with cost $=$ distance^2:

$$I[\nabla P_t] = \min_{s_\# \chi_\Omega = \alpha_t} I[s], \quad I[s] = \int_\Omega |s(x) - x|^2 dx.$$

Solving Semigeostrophic system in Dual Space

$$
\begin{aligned}
&\partial_t \alpha + \mathrm{div}(U\alpha) = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^3, \\
&U(t,X) = J(X - \nabla P_t^*(X)), \\
&\nabla P_{t\#}\chi_\Omega = \alpha_t, \\
&\alpha_{|t=0} = \alpha_0.
\end{aligned}
$$

Existence in dual space: time stepping (Benamou-Brenier, Cullen-Gangbo): let $\Delta t = h$.
Suppose, at time at $t_k = kh$, the $P_k(x)$ convex, and measure $\alpha_k(x)$ are given. Determine velocity

$$U_k(X) = J(X - \nabla P_k^*(X))$$

(plus some regularization...). Solve transport equation

$$
\begin{aligned}
&\partial_t \alpha + \mathrm{div}(U_k\alpha) = 0 \quad \text{in} \quad (kh,(k+1)h) \times \mathbb{R}^3, \\
&\alpha_{|t=kh} = \alpha_k.
\end{aligned}
$$

Solving Semigeostrophic system in Dual Space

Then define $\alpha_{k+1} = \alpha((k+1)h)$.

From $\operatorname{div} U_k = 0$ get $\int \alpha_k dx = \int \alpha_{k+1} dx$.

Determine P_{k+1} by solving Monge-Kantorovich problem: P_{k+1} is convex and ∇P_{k+1} is the optimal map between χ_Ω and α_{k+1}.

Then send h to $0+$. Using convexity, can pass to the limit in equations.

Semigeostrophic system in Physical Space

Let (P, α) be a solution in dual space.
Obtain solution (P, u) in physical space, i.e. define physical velocity u. Formally, use relation $x(t) = \nabla P_t(X(t))$ for particle paths. Differentiate:

$$
\begin{aligned}
u(t,x) =& \partial_t \nabla P_t^*(X) + U \cdot \nabla(\nabla P_t^*(X)) \\
=& \partial_t \nabla P^*(t, \nabla P_t(x)) + D^2 P_t^*(\nabla P_t(x))[J(\nabla P(t,x) - x)],
\end{aligned}
$$

Here P_t, P_t^* are convex, i.e $D^2 P_t^*$ is a measure, and $\nabla P_t \in L^\infty$. Their product is not well-defined.

Semigeostrophic system in Physical Space: Eulerian solutions

Also, (P, u) is a weak (Eulerian) solution of SG if $\mathrm{div} u = 0$ and

$$\int_{(0,T)\times\Omega} \{\nabla P(t,x) \cdot [\partial_t \phi(t,x) + (u(t,x) \cdot \nabla)\phi(t,x)]$$

$$+ J[\nabla P(t,x) - x] \cdot \phi(t,x)\} dtdx + \int_\Omega \nabla P_0(x) \cdot \phi(0,x) dx = 0.$$

for any $\phi \in C_c^1([0,T) \times \Omega; \mathbb{R}^3)$.
Since $\nabla P_t \in L^\infty$, need $u \in L^1_{loc}$.
Existence of $u \in L^1_{loc}$ is not known.
Apriori estimates of u as a measure: Loeper (2005).

Semigeostrophic system in Physical Space: Eulerian solutions

Recent works: De Philippis, Figalli (2011) regularity for Monge-Ampere: if $\Lambda \geq f(x) \geq \lambda > 0$ in Ω and

$$\det D^2 u = f \text{ in } \Omega,$$

then $u \in W^{2,1}(\Omega)$ (and slightly better). Boundary regularity if $\partial\Omega$ is convex and smooth.

Ambrosio, Colombo, De Philippis, Figalli (2011, 2012): existence of Eulerian solutions of SG in 2D-periodic and 3D cases if the density in dual space α_0 is strictly positive +...

In the case when the boundary of the support of α_0 is nonempty, say when supp(α_0) is compact, existence of Eulerian solutions is not known, not clear whether can be expected.

The case when the support of α_0 is compact is physically interesting: related to modeling of front formation in atmospheric flows.

Weak Lagrangian Solutions in Physical Space

Cullen-F. 2006.

If (P, u) is smooth, then define flow map of u:
$F : [0, T] \times \Omega \to \Omega$ satisfying

$$\partial_t F(t, x) = u(t, F(t, x))$$
$$F_{|t=0} = Id.$$

Since $u \cdot \nu = 0$ on $\partial\Omega$, it follows for each $t \geq 0$ that $F_t : \Omega \to \Omega$ is diffeomorphism. Then F determines u.

SG system in terms of (P, F):

$$F_{t\#}\chi_\Omega = \chi_\Omega \quad \text{for all} \quad t > 0,$$
$$F_0 = Id,$$

and $Z(t, x) = \nabla P(t, F_t(x))$ is a solution of the ODE

$$\partial_t Z(t, x) = J[Z(t, x) - F(t, x)] \qquad \text{in} \quad [0, T) \times \Omega,$$
$$Z(0, x) = \nabla P_0(x).$$

Weak Lagrangian Solutions in Physical Space

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set, and $T > 0$. Let $P_0(x) \in W^{1,\infty}(\Omega)$ be convex. Let $r \in [1, \infty)$. Let

$$P \in L^\infty([0,T); W^{1,\infty}(\Omega)) \cap C([0,T); W^{1,r}(\Omega)),$$
$$P_t(\cdot) \quad \text{is convex in} \quad \Omega \quad \text{for each} \quad t \in [0,T).$$

Let $F : [0,T) \times \Omega \to \Omega$ satisfy $F \in C([0,T); L^r(\Omega; \mathbb{R}^3))$. (P, F) is a weak Lagrangian solution of SG system if

$$F_{t\#}\chi_\Omega = \chi_\Omega \quad \text{for all} \quad t > 0,$$
$$F_0 = Id,$$

and $Z(t,x) = \nabla P(t, F_t(x))$ is a weak solution of the ODE

$$\partial_t Z(t,x) = J[Z(t,x) - F(t,x)] \qquad \text{in} \quad [0,T) \times \Omega,$$
$$Z(0,x) = \nabla P_0(x).$$

Existence of Weak Lagrangian Solutions in Physical Space

Cullen, Feldman (2006): if $\alpha_0 := \nabla P_0 \# \chi_\Omega \in L^q$, $q > 1$, for 3D rigid boundaries and 2D shallow water SG models.

Outline of proof: Combining Cullen-Gangbo time-stepping procedure, and Ambrosio theory of Hamilton-Jacobi equations and ODE with BV vector fields, obtain Lagrangian flow $\Phi(t, X)$ in dual space, and

$$\alpha_t = \Phi_{t\#}\alpha_0.$$

Here we use that $U(t, X) = J(X - \nabla P_t^*(X))$ is BV (as a gradient of convex function) and divergence-free (by $J\nabla$-structure).

Then the flow in physical space is

$$F_t = \nabla P_t^* \circ \Phi_t \circ \nabla P_0.$$

Faria, Lopes-Filho, Nussenzveig-Lopes (2009): $q = 1$ case.

Remark on condition $\alpha_0 := \nabla P_0 \# \chi_\Omega \in L^q$

This condition is a form of strict convexity of P_0. For example, if P_0 is uniformly strictly convex, i.e. $P_0(x) - \varepsilon x^2$ is convex, then $\alpha_0 \in L^\infty$.
If P_0 is affine on a set of positive measure, then α_0 has a delta-function (i.e. a point of nonzero measure).

Case of $\alpha_0 := \nabla P_0 \# \chi_\Omega$ is a measure

This case is physically relevant.

Solutions in dual space Loeper (2006), Ambrosio, Gangbo (2008): P_t is convex and α_t is a measure, and satisfy:

$$
\begin{aligned}
&\partial_t \alpha + \mathrm{div}(U\alpha) = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^3, \\
&U(t,X) = J(X - \bar{\gamma}_t(X)), \\
&\nabla P_{t\#} \chi_\Omega = \alpha_t, \\
&\alpha_{|t=0} = \alpha_0,
\end{aligned}
$$

where $\bar{\gamma}_t(X)$ is the barycentric projection of the optimal Kantorovich plan $\gamma_t := \left(\nabla P_t \times \mathrm{Id}\right)_{\#} \chi$ having α_t and χ as first and second marginals, respectively. It is defined by

$$
\int_{\mathbb{R}^3} \xi(X) \cdot \bar{\gamma}_t(X) d\alpha_t(X) = \iint_{\mathbb{R}^3 \times \Omega} \xi(X) \cdot y d\gamma_t(X,y)
$$

for all continuous $\xi : \mathbb{R}^3 \to \mathbb{R}^3$ of at most quadratic growth.

Case $\alpha_0 := \nabla P_0 \# \chi_\Omega$ is a measure: Flow map in physical space

(Tudorascu-F. 2012)
Define Lagrangian solutions in physical space when α_0 is singular: Since $\bar{\gamma}_t$ replaces ∇P_t^*, try

$$F_t = \bar{\gamma}_t \circ \Phi_t \circ \nabla P_0.$$

Example: $\Omega = B_1$, $P_0(x) = 0$.
Then $P_t(x) = 0$ on B_1 for all t, and $\alpha_t = \delta_0$. Also $P_t^*(X) = X$.
Thus $\bar{\gamma}_t(0) = 0$ which defined $\bar{\gamma}_t(X)$ for α_t-a.e. $X \in bR^3$.
Can set $\bar{\gamma}_t(X) = \nabla P_t^*$ for $X \neq 0$.
Also, $U(t, X) = J(X - \bar{\gamma}_t(X))$, thus $\Phi(t, 0) = 0$ is a solution of ODE $\frac{d}{dt}\Phi(t, 0) = U(t, U(t, \Phi(t, 0))$. In fact, this is a continuous extension to $X = 0$ of the regular flow $\Phi(t, X)$ for vector field $U(t, X) = J(X - \nabla P_t^*)$.

Case $\alpha_0 := \nabla P_0 \# \chi_\Omega$ is a measure: Flow map in physical space

We get $F_t(x) = \bar{\gamma}_t \circ \Phi_t \circ \nabla P_0(x) = 0$ for any $x \in B_1$, $t > 0$.
In particular, $F_{t\#}\chi_\Omega = \delta_0 \neq \chi_\Omega$.
Also can show: for regularizations $P_0^\varepsilon = \varepsilon \| x|^2$ get $F^\varepsilon \rightharpoonup F$ weakly-* in $L^\infty([0,T] \times B_1)$, but not in $L^p(B_1)$ for each t.

$F_t = \bar{\gamma}_t \circ \Phi_t \circ \nabla P_0$. Issues to address:
(i) If Φ_t is a Lagrangian flow ∇P_t^* (or, equivalently, for $\bar{\gamma}_t$), then it is not clear if $\alpha_t = \Phi_{t\#}\alpha_0$ holds (or even well-defined);
(ii) If $\alpha_t = \nabla P_{t\#}\chi_\Omega$ is a singular measure, then $(\bar{\gamma}_t \circ \nabla P_t)_\#\chi_\Omega \neq \chi_\Omega$. Thus $F_0\#\chi_\Omega \neq \chi_\Omega$. Then probably $F_{t\#}\chi_\Omega \neq \chi_\Omega$ for $t > 0$. Instead, define "reduced domain" measures $\mu_t = (\bar{\gamma}_t \circ \nabla P_t)_\#\chi_\Omega$. Then $F_{t\#}\chi_\Omega = \mu_t$, and $F_{t\#}\mu_0 = \mu_t$. Note: if $\alpha_t \in L^1(\mathbb{R}^3)$, then $\mu_t = \chi_\Omega$.

Case $\alpha_0 := \nabla P_0 \# \chi_\Omega$ is a measure: Flow map in physical space

(iii) Even if $\alpha_t = \Phi_{t\#}\alpha_0$, the continuity in time $F \in C([0,T); L^r(\Omega;\mathbb{R}^3))$ is unlikely to hold if α_t are singular measures.
We can prove weaker continuity of $t \to F_t(\cdot)$, related to ∇P_t: for any $\phi \in C^1_c(\mathbb{R}^3;\mathbb{R}^3)$

$$\lim_{t\to t_0}\int_\Omega \phi(\nabla P_{t_0}\circ F_{t_0}(x))\cdot F_t(x)dx = \int_\Omega \phi(\nabla P_{t_0}\circ F_{t_0}(x))\cdot F_{t_0}(x)dx,$$

in particular

$$\lim_{t\to 0^+}\int_\Omega \phi(\nabla P_0(x)) \cdot F_t(x)dx = \int_\Omega \phi(\nabla P_0(x)) \cdot x dx.$$

Call this P-continuity.

Case $\alpha_0 := \nabla P_0 \# \chi_\Omega$ is a measure: Lagrangian Solutions in Physical Space

Let $P_0(x) \in W^{1,\infty}(\Omega))$ be convex. Let $r \in [1, \infty)$. Let

$$P \in C([0,T); W^{1,r}(\Omega)),$$
$$P_t(\cdot) \quad \text{is convex in} \quad \Omega \quad \text{for each} \quad t \in [0,T).$$

Let $F : [0,T) \times \Omega \to \Omega$ be P-continuous.
(P, F) is a weak Lagrangian solution of SG system if

$$F_{t\#}\chi_\Omega = \mu_t \quad \text{and} \quad F_{t\#}\mu_0 = \mu_t \quad \text{for all} \quad t \geq 0,$$
$$\text{where} \quad \mu_t = (\bar{\gamma}_t \circ \nabla P_t)_{\#}\chi_\Omega,$$

and $Z(t,x) = \nabla P(t, F_t(x))$ is a weak solution of the ODE

$$\partial_t Z(t,x) = J[Z(t,x) - F(t,x)] \qquad \text{in} \quad [0,T) \times \Omega,$$
$$Z(0,x) = \nabla P_0(x).$$

Case $\alpha_0 := \nabla P_0 \# \chi_\Omega$ is a measure: Existence of Lagrangian Solutions in Physical Space

Proposition Given solution (P_t, α_t) in dual space: If there exists Lagrangian flow in dual space $\Phi(t, X)$ satisfying

$$\partial_t \Phi(t, X) = J(\Phi(t, X) - \bar{\gamma}_t(\Phi(t, X))), \quad \Phi_{|t=0} = Id,$$
$$\alpha_t = \Phi_{t\#}\alpha_0,$$

then (P, F) with $F_t = \bar{\gamma}_t \circ \Phi_t \circ \nabla P_0$, is a Lagrangian solution in physical space.

Theorem If $P_0 = \max_{k=1,\dots,n} L_k(X)$, where each L_k is an affine function, then there exists a Lagrangian soluiton (P, F) in physical space.

Remark In the conditions of theorem, α_0 is a convex combination of delta-functions.

Properties of Weak Lagrangian Solutions in Physical Space

Geostrophic energy:

$$E(t) = \int_\Omega |\nabla P_t(x) - x|^2 dx.$$

Formally $E(t) = const$ on solutions of SG system.

Theorem, If (P, F) is a weak Lagrangian solution. Then

- Let $\alpha_t := \nabla P_{t\#}\chi_\Omega$. Then (P, α) is a solution of SG in dual space:

$$\partial_t \alpha + \mathrm{div}(U\alpha) = 0 \quad \text{in} \quad (0, \infty) \times \mathbb{R}^3,$$
$$U(t, X) = J(X - \bar{\gamma}_t(X)).$$

- $E(t) = const.$

Remarks: (i) $\alpha_t = \nabla P_{t\#}\chi_\Omega$ may be a singular measure;
(ii) Once we know (P, α) is a solution, then $E(t) = const$ follows from work of Ambrosio-Gangbo.

Relaxed Lagrangian solutions in physical space

(Tudorascu-F., 2013)
Existence for arbitrary (possibly non-strictly) convex initial P_0:
replace flow map $F_t : \Omega \to \Omega$ by transport plan σ_t on $\Omega \times \Omega$.

Let $\alpha_0 = \nabla P_{0\#}\chi_\Omega \in L^q$ and (P_t, F_t) is a Lagrangian solution.
Define measure $\sigma_t = (Id \times F_t)_\# \chi_\Omega$ on $\Omega \times \Omega$. Then:
(i) $\sigma_0 = (Id \times Id)_\# \chi_\Omega$;
(ii) $\pi_{1\#}\sigma_t = \chi_\Omega$, $\pi_{2\#}\sigma_t = \chi_\Omega$, where $\pi_k(\mathbf{x}) = \mathbf{x}_k$ for $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \Omega \times \Omega$, $k = 1, 2$.
(iii) for any $\varphi \in C_c^1([0,T) \times \Omega;\, \mathbb{R}^3)$

$$\int_0^T \int_{\Omega\times\Omega} [\nabla P_t(y) \cdot \partial_t \varphi(t,x) + J(\nabla P_t(y) - y) \cdot \varphi(t,x)]\, d\sigma_t(x,y)\, dt$$
$$+ \int_\Omega \nabla P_0(x) \cdot \varphi(0,x)\, dx = 0.$$

To define relaxed solutions, note another property of (P_t, F_t) when $\alpha_t \in L^q$:

Renormalized Lagrangian solutions

If $\alpha_0 = \nabla P_{0\#}\chi_\Omega \in L^q$, $q > 1$ and $P_0 \in W^{1,\infty}(\Omega)$, then Lagrangian solutions (P, F) satisfy: $Z, \partial_t Z \in L^\infty([0,T) \times \Omega)$ and

$$\partial_t Z(t,x) = J[Z(t,x) - F(t,x)] \text{ a.e. in } [0,T) \times \Omega.$$

Thus if $\xi \in C^1(\mathbb{R}^3)$

$$\partial_t(\xi(Z(t,x)) = \nabla\xi(Z(t,x)) \cdot J[Z(t,x) - F(t,x)] \quad \text{a.e. in } [0,T) \times \Omega.$$

Thus for any $\xi \in C^1_b(\mathbb{R}^3)$, $\varphi \in C^1_c([0,T) \times \Omega)$

$$\int_0^T \int_\Omega \Big[\xi(Z(t,x))\partial_t\varphi(t,x) + \nabla\xi(Z(t,x)) \cdot J\Big(Z(t,x) - F(t,x)\Big)\varphi(t,x)\Big]\,dx\,dt + \int_\Omega \xi(\nabla P_0(x)\varphi(0,x))\,dx = 0.$$

Renormalized Relaxed Lagrangian solutions

Definition. Let P_0 be convex on Ω. Let $P : [0,T) \times \Omega \to \mathbb{R}^1$, and let $\sigma = \int_0^T \sigma_t dt$ be a Borel measure on $[0,T) \times \Omega \times \Omega$.

Then (P, σ) is a renormalized relaxed Lagrangian solution of SG with initial data P_0 if

(i) P_t is convex for each $t \in [0,T)$,

(ii) $\pi_{1\#}\sigma_t = \chi_\Omega$, $\pi_{2\#}\sigma_t = \chi_\Omega$, where $\pi_k(\mathbf{x}) = \mathbf{x}_k$ for $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \Omega \times \Omega$, $k = 1, 2$.

(iii) for any $\xi \in C_b^1(\mathbb{R}^3)$, $\varphi \in C_c^1([0,T) \times \Omega)$

$$\int_0^T \int_{\Omega\times\Omega} \Big[\xi(\nabla P_t(y))\partial_t\varphi(t,x) + \nabla\xi(\nabla P_t(y)) \cdot J\Big(\nabla P_t(y) - y\Big)\varphi(t,x)\Big] d\sigma_t(x,y)\, dt + \int_\Omega \xi(\nabla P_0(x))\varphi(0,x)\, dx = 0.$$

Equation (iii) well-defined for $\nabla P \in L^\infty([0,T); L^1(\Omega))$ by (ii).

Renormalized Relaxed Lagrangian solutions

These solutions are somewhat underdeterminate: heuristically, $\sigma_t(x,y)$ can be modified on flat parts of P_0 and P_t resp.

Example: $\Omega = B_1$, $P_0(x) = 0$. Then $P_t(x) = 0$ for all t. Equation is: for any $\xi \in C^1_b(\mathbb{R}^3)$, $\varphi \in C^1_c([0,T) \times \Omega)$

$$\int_0^T \int_{\Omega\times\Omega} \Big[\xi(0)\partial_t\varphi(t,x) - \varphi(t,x)\nabla\xi(0)\cdot Jy\Big] d\sigma_t(x,y)\, dt + \int_\Omega \xi(0)\varphi(0,x)\, dx = 0,$$

and using $\pi_{1\#}\sigma_t = \chi_\Omega$,

$$\int_0^T \int_{\Omega\times\Omega} \varphi(t,x)\nabla\xi(0)\cdot Jy\, d\sigma_t(x,y)\, dt = 0.$$

Thus any Borel family $\sigma_t(x,y)$ with marginals χ_Ω and y-barycenter $\bar\sigma_t(x) = 0$ works.

Such underdeterminacy is physically natural/expected.

Renormalized Relaxed Lagrangian solutions

Existence and stability. Let $P_0 \in W^{1,2}(\Omega)$ be convex. Then there exists a renormalized relaxed Lagrangian solution with initial data P_0.
Moreover, if $q \in (1, \infty]$, and P_k^0 is convex in Ω and $\alpha_0^k := \nabla P_k^{0\#} \chi_\Omega \in L^q(\Omega)$ for $k = 1, \dots,$ with $\alpha_0^k \rightharpoonup \alpha_0$ weakly in $\mathcal{M}(\Omega)$ as $k \to \infty$, then, denoting (P^k, F^k) Lagrangian solution for initial data P_0^k, and $\sigma_t^k := (Id \times \nabla P_t^k)_\# \chi_\Omega$ $\sigma^k = \int_0^T \sigma_t^k dt$, and selecting a subsequence, get

$$\nabla P_t^k \to \nabla P_t \quad \text{in} \quad L^2(\Omega) \quad \text{for all} \quad t \in [0, T);$$
$$\alpha_t^k \rightharpoonup \alpha_t \text{ weakly in } \mathcal{M}(\Omega) \quad \text{for all} \quad t \in [0, T);$$
$$\sigma^k \rightharpoonup \sigma \text{ weakly in } \mathcal{M}([0, T) \times \Omega \times \Omega),$$

and (P, σ) is a renormalized relaxed Lagrangian solution. Also, (P, α) is a distributional solution in dual space.

Renormalized Relaxed Lagrangian solutions

Return to dual space, conservation of geostrophic energy. Let (P, σ) be a renormalized relaxed Lagrangian solution and $\alpha_t = \nabla P_{t\#}\chi_\Omega$. Then (P, α) is a distribution solution in dual space. In particular geostrophic energy is conserved.

Renormalization property is used in this proof: for test function $\varphi(t)\xi(X)$ in dual space, with $\varphi(0) = 0$ for simplicity:

$$\int_0^T \int_{\mathbb{R}^3} \varphi'(t)\xi(X)d\alpha_t dt = \int_0^T \int_\Omega \varphi'(t)\xi(\nabla P_t(y))dydt$$

$$= \int_0^T \int_{\Omega\times\Omega} \varphi'(t)\xi(\nabla P_t(y))d\sigma_t(x,y)dt$$

$$= -\int_0^T \int_{\Omega\times\Omega} \varphi(t)\nabla\xi(\nabla P_t(y)) \cdot J\Big(\nabla P_t(y) - y\Big)d\sigma_t(x,y)\,dt$$

$$= -\int_0^T \int_\Omega \varphi(t)\nabla\xi(\nabla P_t(y)) \cdot J\Big(\nabla P_t(y) - y\Big)dy\,dt = -I_1 - I_2.$$

Renormalized Relaxed Lagrangian solutions

$$I_1 = \int_0^T \int_\Omega \varphi(t)\nabla\xi(\nabla P_t(y)) \cdot J\nabla P_t(y) dy\, dt$$
$$= \int_0^T \int_{\mathbb{R}^3} \varphi(t)\nabla\xi(X) \cdot JX\, d\alpha_t(X)\, dt$$

Denote $\gamma_t = (Id \times \nabla P_t)_{\#}\chi_\Omega$:

$$I_2 = -\int_0^T \int_\Omega \varphi(t)\nabla\xi(\nabla P_t(y)) \cdot Jy\, dy\, dt$$
$$= -\int_0^T \int_{\Omega\times\mathbb{R}^3} \varphi(t)\nabla\xi(X) \cdot Jy\, d\gamma_t(y, X)\, dt$$
$$= -\int_0^T \int_{\mathbb{R}^3} \varphi(t)\nabla\xi(X) \cdot J\bar{\gamma}_t(X)\, d\alpha_t(X)\, dt$$

Get: $\int_0^T \int_{\mathbb{R}^3} [\partial_t\zeta + \nabla\zeta \cdot U] d\alpha_t(X) dt = 0$, for
$U(t, X) = J(X - \bar{\gamma}_t(X)), \quad \zeta(t, x) = \varphi(t)\xi(X)$.

Renormalized Relaxed Lagrangian solutions

Continuity in time. Let (P, σ) be a RRL solution with $\nabla P_0 \in L^2(\Omega)$. Then, on $[0, T]$:

Equation in dual space $\Rightarrow$
$t \to \alpha_t$ continuous in Wasserstein W_2 metric;

$t \to \nabla P_t$ is continuous in $L^2(\Omega)$ with $(\nabla P_t)_{|t=0} = \nabla P_0$;

Define $G_t : \Omega \times \Omega \to \mathbb{R}^3 \times \mathbb{R}^3$ by $G_t(x, y) = (x, \nabla P_t(y))$, then $t \to G_{t\#}\sigma_t$ is continuous with respect to narrow convergence on $\Omega \times \mathbb{R}^3$: From equation, for any $\psi, \xi \in C_c^1(\mathbb{R}^3)$

$$t \to \int_{\Omega\times\Omega} \psi(x)\, \xi(\nabla P_t(x))\, d\sigma_t(x, y) = \int_{\Omega\times\mathbb{R}^3} \psi(x)\, \xi(Y)\, d(G_{t\#}\sigma_t)(x, y)$$

is continuous.

Also, for any $s \in [0, T]$: $t \to G_{s\#}\sigma_t$ is continuous at $t = s$ with respect to narrow convergence on $\Omega \times \mathbb{R}^3$.
Initial condition hold: $G_{0\#}\sigma_0 = G_{0\#}\delta_{\{x=y\}}$, where $\delta_{\{x=y\}} := (Id \times Id)_{\#}\chi_\Omega$.

Open problems

- Uniqueness of weak (renormalized) solutions. Possibly weak-strong uniqueness.
- Existence of solutions for the case of variable Coriolis parameter: dual space is not defined.