3. Existence of Distributional solutions to the semigeostrophic equations. Maria,

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①. The equation

$\Omega \subseteq \mathbb{R}^3$ convex

$$(IE)\quad \begin{cases} \partial_t v_t + v_t \nabla v_t = -\nabla p_t + \underline{\text{ext forces}} \\ \partial_t m_t + v_t \nabla m_t = 0 \\ \nabla v_t = 0 . \end{cases} \quad \text{in } \Omega \times [0,\infty)$$

where $v_t : [0,\infty) \times \Omega \to \mathbb{R}^3$ · velocity

$p_t : [0,\infty) \times \Omega \to \mathbb{R}$ pressure.

$m_t : [0,\infty) \times \Omega \to \mathbb{R}$ density.

take ext forces to be $-Jv_t - \begin{pmatrix} 0 \\ 0 \\ m_t \end{pmatrix}$.

where $J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

v_t^g = semigeostrophic wind $= J\nabla p_t$.

$\partial_3 p_t = m_t$.

$$\partial_t u_t^g + u_t \cdot \nabla u_t^g = -\nabla p_t - J u_t - \begin{pmatrix} 0 \\ 0 \\ m_t \end{pmatrix}$$

Energetic considerations $\Longrightarrow$ $P_t(x) = p_t(x) + \frac{x_1^2 + x_2^2}{2}$ convex

$$(SG) \begin{cases} \cancel{\partial_t \nabla P}\ \partial_t \nabla P_t + \nabla^2 P_t\, u_t = J(\nabla P_t - x). \\ \nabla u_t = 0. \end{cases}$$

$P_t(\cdot)$ convex $\langle u_t, \nu_\Omega \rangle = 0$. $\quad u_t \cdot \underset{\text{normal to } \partial\Omega}{\nu_\Omega} = 0$

P_0 given.

Dual equation

$$\rho_t = \nabla P_t \# \mathcal{L}_\Omega .$$

$$\frac{d}{dt} \int \varphi(x)\, d\rho_t(x) = \frac{d}{dt} \int \varphi(\nabla P_t)\, dx = \int \nabla\varphi(\nabla P_t) \cdot \partial_t(\nabla P_t)\, dx .$$

$$= -\int \underbrace{\nabla \varphi(\nabla P_t)\, \nabla^2 P_t}_{\nabla(\varphi(\nabla P_t))} \underset{\text{inner product}}{\cdot} u_t + \int \nabla\varphi(\nabla P_t)\, J(\nabla P_t - x)\, dx .$$

$$= \int \nabla \varphi \ \underbrace{J(x - \nabla P_t^*)}_{u_t(x)}\, d\rho_t .$$

where ∇P_t^* is the convex conjugate of ∇P_t.

Thus we get.

$$(\text{dual-SG}) \quad \begin{cases} \partial_t \rho_t + \nabla \cdot (u_t \rho_t) = 0. \\ \rho_t = \nabla P_t \# \mathcal{L}_\Omega . \ (\Leftrightarrow \det(\nabla^2 P_t^*) = \rho_t) \\ u_t(x) = J(x - \nabla P_t^*(x)). \end{cases}$$

Remarks.

① ∇P_t is the OT map : $\mathcal{L}_{\Omega} \to \rho_t$.

② There is no v_t.

③ Nice continuity eq. with $\nabla u_t = 0$

④ Istantaneous coupling between ρ_t and u_t.

⑤ Analogy with Euler in ~~Voscous~~ vorticity $(\mathbb{R}^2)$

$$\begin{cases} \partial_t \omega_t + \nabla(v_t \omega_t) = 0. \\ \omega_t = \Delta \chi_t. \\ v_t = J \nabla \chi_t. \end{cases} \qquad \omega_t = \mathrm{curl}(v_t)$$

<u>Thm</u> ρ_0 compact supp. $\Rightarrow \exists (\rho_t, P_t)$, solution to (dual-SG).

<u>Remark</u>, uniqueness for (dual-SG) is open.

From dual to physical solution (original ~~solution~~ eqn.)

P_t sol for (dual - SG)

$u_t = \partial_t \nabla P_t^*(\nabla P_t) + \nabla^2 P_t^*(\nabla P_t) J(\nabla P_t - x)$. (**)

Problems ① $\nabla^2 P_t^*$ is a measure.

② time regularity for ∇P_t^* : $\rho_t \to \mathcal{L}_\Omega$

Thm 2. P_0 convex s.t $\rho_0 = \nabla P_0 \# \mathcal{L}_\Omega$

$\rho_0, \frac{1}{\rho_0} \in \mathcal{L}^\infty_{loc}(\mathbb{R}^3)$. $\rho_0(x) \le \frac{c}{1+|x|^k}$, $k>0$

Let P_t sol for (dual - SG) and u_t as (**).

Then (P_t, u_t) is a distributional sol to (SG).

Space regularity.

Theorem 3. P be a solution of $\det D^2 P = f$.

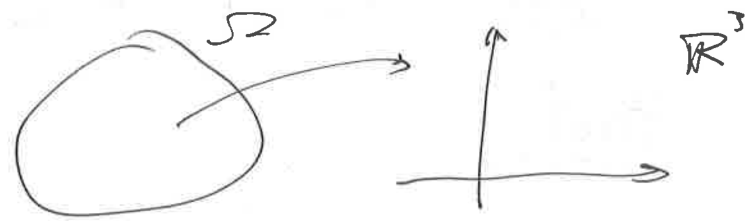
with $f, \frac{1}{f} \in \mathcal{L}^\infty_{loc}(\mathbb{R}^3)$

Then $P \in W^{2,\gamma}(B_R)$ for some $\gamma = \gamma(R) > 1$

In this problem $\det \partial^2 P_t^* = \rho_t \Rightarrow P_t^* \in W^{2,\gamma}$.

Time regularity:

Idea: U_t and $\partial_t \nabla P_t(\nabla P_t^{*})$ are admissible velocity fields for ρ_t.



$$X_t(y) = \nabla P_t(\nabla P_0^*(y))$$

$$\partial_t X_t(y) = \partial_t \nabla P_t(\nabla P_0^*(y)).$$

$$= \underbrace{\partial_t \nabla P_t(\nabla P_t^*}(X_t(y))).$$

$X_t \# \rho_0$ is a sol. to $\boxed{(CE)}$ with $\partial_t \nabla P_t(\nabla P_t^*)$ — the continuity equation for the velocity field $\partial_t \nabla P_t(\nabla P_t^*)$

$$X_t \# \rho_0 = \nabla P_t \# \alpha = \rho_t$$

$$\partial_t \rho_t = -\nabla\cdot(U_t \rho_t).$$

$$\parallel$$

$$\partial_t(\det \nabla^2 P^*) = -\nabla\left((\nabla^2 P^*)^{-1}\cdot \partial_t \nabla P_t^* \rho_t\right)$$

multiply by $\partial_t P_t^*$, at both side, then take the integration.

$$\int \rho_t \left|\partial_t \nabla P_t^* \cdot (\nabla^2 P_t^*)^{-\frac{1}{2}}\right|^2 = \int \rho_t U_t \cdot \partial_t \nabla P_t^*$$

$$\le \left(\int \rho_t |U_t|^2 |\nabla P_t^*|^2\right)^{1/2} \cdot (LHS)^{\frac{1}{2}}.$$

Fundamental estimate. $\int \rho_t |\partial_t \nabla P_t^* \nabla^2 P_t^{-\frac{1}{2}}|^2 \le \int \rho_t |U_t|^2 |\nabla P_t^*|$

Prop: ρ_t sol. $\partial_t \rho_t + \nabla\cdot(u_t \rho_t) = 0$.

Let $\nabla P_t^* : \rho_t \to \Omega$

Then $\partial_t \nabla P_t^* \in L^{\frac{2r}{1+r}}(B_R)$.

Proof.

$$\left(\int \rho_t \left|\partial_t \nabla P_t^*\right|^{\frac{2r}{1+r}}\right)^{\frac{1+r}{2r}}$$

$$= |\nabla^2 P_t^*|^{\frac{1}{2}} (\partial_t \nabla P_t^*)(\nabla^2 P_t^*)^{-\frac{1}{2}}$$

$$\leq \left(\int \rho_t |\nabla^2 P_t^*|^r\right)^{\frac{1}{2r}} \left(\int \rho_t \left|(\nabla^2 P_t^*)^{-\frac{1}{2}} \partial_t \nabla P_t^*\right|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\int \rho_t |\nabla^2 P_t^*|^r\right)^{\frac{1}{2r}} \left(\int \rho_t |u_t|^2 |\nabla^2 P_t^*|\right)^{\frac{1}{2}}.$$

$\leq \ldots$

Proof of Thm 2 Test (dual SG) with $y\,\phi(\nabla P_t^*(y))$.

for any $\phi \in C_c^\infty(\Omega \times [0,\infty))$.

Legrangian point of view

Def: $b : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is Borel velocity field. $X : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a RLF if.

① $X_t(x) = x + \int_0^t b_s(X_s(x))\,ds$ for a.e. x.

② $X_t \# \mathcal{L} \le C\mathcal{L}$

Remark: $b \in W^{1,1}$ (or BV), $|\nabla \cdot b| \le C \Rightarrow \exists!\, X$.

Thm 4 $\exists$, RLF for v_t.

Proof. $y_t(\ )$ be the RLF for U_t.

$$X_t = \nabla P_1 (y_t (\nabla P_0^*))$$

Open problems:

① Uniqueness of the RLF in physical problem
② Uniqueness of v_t.
③ Existence of smooth solutions starting from smooth data
④ Existence for ρ_0 compactly supp. in the physical variables