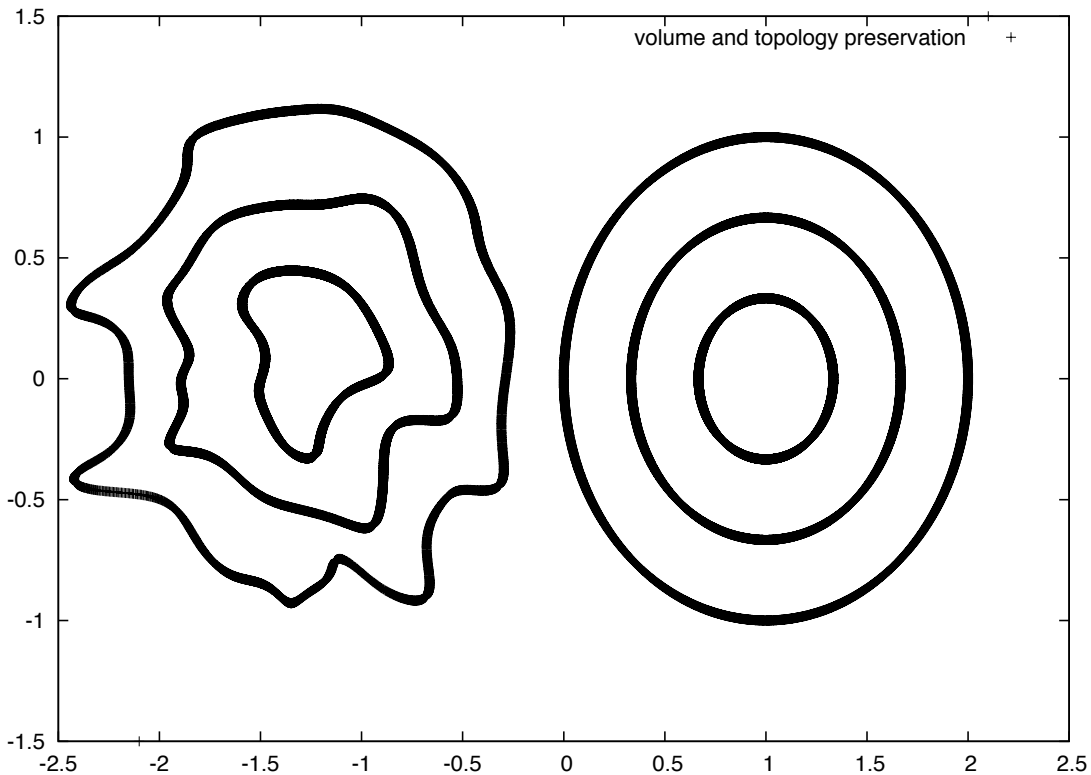
Level-set volume-preserving diffusions

Yann Brenier
CNRS, Centre de mathématiques Laurent Schwartz
Ecole Polytechnique, FR 91128 Palaiseau

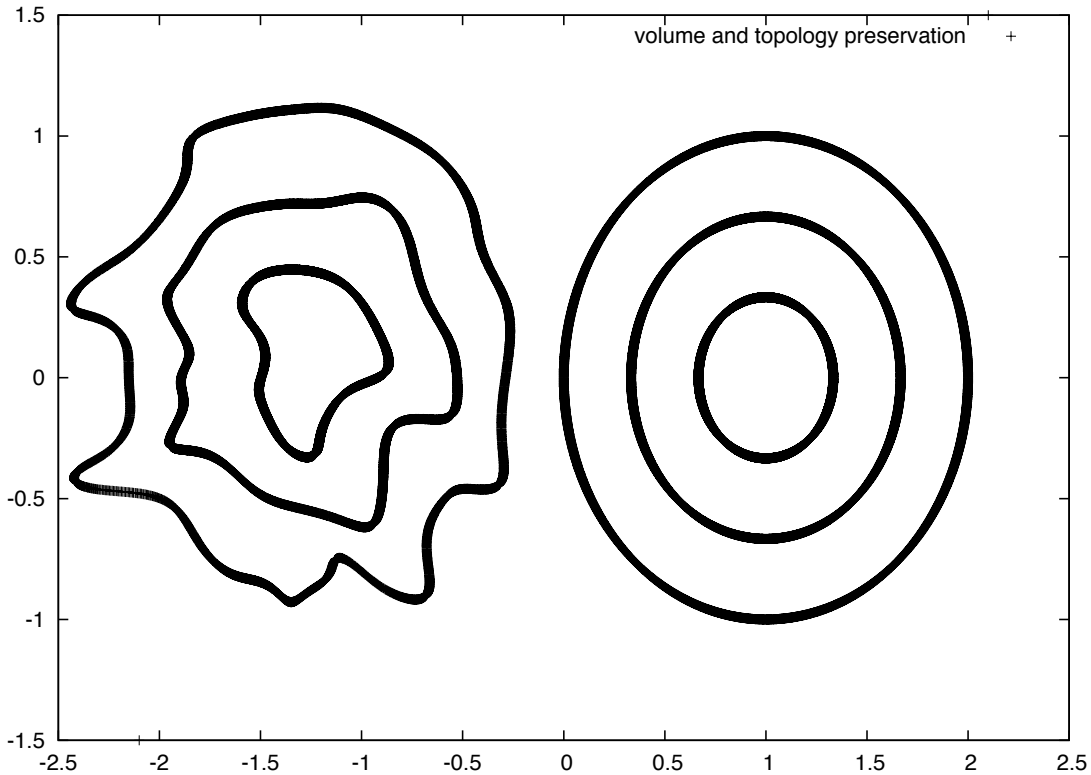
Fluid Mechanics, Hamiltonian Dynamics, and Numerical Aspects of Optimal Transportation,
MSRI, October 14-18, 2013

EQUIVALENCE OF TWO DIFFERENT FUNCTIONS WITH LEVEL SETS OF EQUAL VOLUME

$$\varphi \sim \varphi_0$$



MOTIVATION: MINIMIZATION PROBLEMS WITH VOLUME CONSTRAINTS ON LEVEL SETS



This goes back to Kelvin. See Th. B. Benjamin, G. Burton etc....

An example in fluid mechanics

Here D $= \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ is the flat torus and φ_0 is a given function on D. We want to minimize the Dirichlet integral among all $\varphi \sim \varphi_0$.

An example in fluid mechanics

Here D $= \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ is the flat torus and φ_0 is a given function on D. We want to minimize the Dirichlet integral among all $\varphi \sim \varphi_0$. This can be rephrased as

$$\inf_{\varphi : D \to \mathbb{R}} \quad \sup_{F : \mathbb{R} \to \mathbb{R}} \quad \frac{1}{2} \int_D |\nabla \varphi(x)|^2 dx + \int_D [F(\varphi(x)) - F(\varphi_0(x))] dx$$

An example in fluid mechanics

Here D $= \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ is the flat torus and φ_0 is a given function on D. We want to minimize the Dirichlet integral among all $\varphi \sim \varphi_0$. This can be rephrased as

$$\inf_{\varphi:D\to\mathbb{R}} \sup_{F:\mathbb{R}\to\mathbb{R}} \frac{1}{2}\int_D |\nabla\varphi(x)|^2 dx + \int_D [F(\varphi(x)) - F(\varphi_0(x))]dx$$

Optimal solutions are formally solutions to

$$-\triangle\varphi + F'(\varphi) = 0$$

for some function F : $\mathbb{R} \to \mathbb{R}$

An example in fluid mechanics

Here D $= \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ is the flat torus and φ_0 is a given function on D. We want to minimize the Dirichlet integral among all $\varphi \sim \varphi_0$. This can be rephrased as

$$\inf_{\varphi:D\to\mathbb{R}} \sup_{F:\mathbb{R}\to\mathbb{R}} \frac{1}{2}\int_D |\nabla\varphi(x)|^2 dx + \int_D [F(\varphi(x)) - F(\varphi_0(x))]dx$$

Optimal solutions are formally solutions to

$$-\triangle\varphi + F'(\varphi) = 0$$

for some function F : $\mathbb{R} \to \mathbb{R}$, and, in 2d, are just stationary solutions to the Euler equations of incompressible fluids.

Discrete version: a NP problem in combinatorics

After discretizing the Dirichlet integral on a lattice with N grid points A_i, we have to find a permutation σ that achieves

$$\inf_{\sigma} \quad \sum_{i,j=1}^{N} c_{\sigma_i \sigma_j} \lambda_{ij}$$

where λ is a matrix that depends on the lattice and

$$c_{ij} = |\varphi_0(A_i) - \varphi_0(A_j)|^2$$

Discrete version: a NP problem in combinatorics

After discretizing the Dirichlet integral on a lattice with N grid points A_i, we have to find a permutation σ that achieves

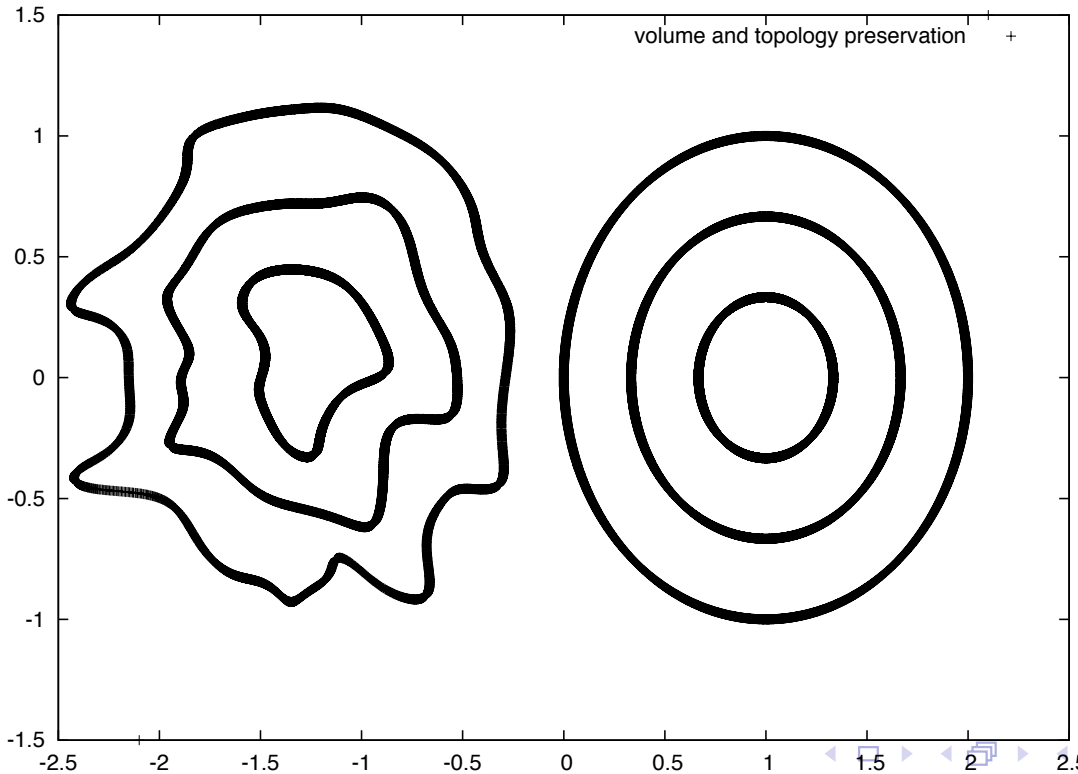
$$\inf_{\sigma} \quad \sum_{i,j=1}^{N} c_{\sigma_i \sigma_j} \lambda_{ij}$$

where λ is a matrix that depends on the lattice and

$$c_{ij} = |\varphi_0(A_i) - \varphi_0(A_j)|^2$$

This is a so-called "quadratic assignment problem", a well known NP problem of combinatorial optimization (also related to the works of F. Memoli and K.T. Sturm, recently presented at MSRI).

THIS SUGGESTS THE CONSTRUCTION OF "GRADIENT FLOWS" OF THE DIRICHLET INTEGRAL (i.e. DIFFUSION EQUATIONS) THAT ARE "LSVP": THEY PRESERVE THE VOLUME OF ALL LEVEL SETS



Transport of functions by divergence-free vector fields

A canonical way to preserve the volume of each level set of a time-dependent scalar function $(\mathbf{t}, \mathbf{x}) \to \varphi(\mathbf{t}, \mathbf{x}) \in \mathbb{R}$ is the transport by a time-dependent divergence-free velocity field $\mathbf{v}(\mathbf{t}, \mathbf{x}) \in \mathbb{R}^{\mathbf{d}}$:

$$\partial_{\mathbf{t}} \varphi + \nabla \cdot (\mathbf{v} \varphi) = \mathbf{0}, \quad \nabla \cdot \mathbf{v} = \mathbf{0}$$

Transport of functions by divergence-free vector fields

A canonical way to preserve the volume of each level set of a time-dependent scalar function $(\mathbf{t}, \mathbf{x}) \to \varphi(\mathbf{t}, \mathbf{x}) \in \mathbb{R}$ is the transport by a time-dependent divergence-free velocity field $\mathbf{v}(\mathbf{t}, \mathbf{x}) \in \mathbb{R}^{\mathbf{d}}$:

$$\boxed{\partial_{\mathbf{t}} \varphi + \nabla \cdot (\mathbf{v} \varphi) = \mathbf{0}, \quad \nabla \cdot \mathbf{v} = \mathbf{0}}$$

If v is smooth enough, this just means $\boxed{\varphi(\mathbf{t}, \xi(\mathbf{t}, \mathbf{x})) = \varphi_{\mathbf{0}}(\mathbf{x})}$

Transport of functions by divergence-free vector fields

A canonical way to preserve the volume of each level set of a time-dependent scalar function $(\mathbf{t}, \mathbf{x}) \to \varphi(\mathbf{t}, \mathbf{x}) \in \mathbb{R}$ is the transport by a time-dependent divergence-free velocity field $\mathbf{v}(\mathbf{t}, \mathbf{x}) \in \mathbb{R}^{\mathbf{d}}$:

$$\partial_{\mathbf{t}} \varphi + \nabla \cdot (\mathbf{v} \varphi) = \mathbf{0}, \quad \nabla \cdot \mathbf{v} = \mathbf{0}$$

If v is smooth enough, this just means $\varphi(\mathbf{t}, \xi(\mathbf{t}, \mathbf{x})) = \varphi_{\mathbf{0}}(\mathbf{x})$
where ξ is the time-dependent family of volume and orientation-preserving diffeomorphisms defined by

$$\partial_{\mathbf{t}} \xi(\mathbf{t}, \mathbf{x}) = \mathbf{v}(\mathbf{t}, \xi(\mathbf{t}, \mathbf{x})), \quad \xi(\mathbf{0}, \mathbf{x}) = \mathbf{x}$$

Transport of functions by divergence-free vector fields

A canonical way to preserve the volume of each level set of a time-dependent scalar function $(\mathbf{t}, \mathbf{x}) \to \varphi(\mathbf{t}, \mathbf{x}) \in \mathbb{R}$ is the transport by a time-dependent divergence-free velocity field $\mathbf{v}(\mathbf{t}, \mathbf{x}) \in \mathbb{R}^{\mathbf{d}}$:

$$\partial_{\mathbf{t}} \varphi + \nabla \cdot (\mathbf{v} \varphi) = \mathbf{0}, \quad \nabla \cdot \mathbf{v} = \mathbf{0}$$

If v is smooth enough, this just means $\varphi(\mathbf{t}, \xi(\mathbf{t}, \mathbf{x})) = \varphi_{\mathbf{0}}(\mathbf{x})$ **where ξ is the time-dependent family of volume and orientation-preserving diffeomorphisms defined by**

$$\partial_{\mathbf{t}} \xi(\mathbf{t}, \mathbf{x}) = \mathbf{v}(\mathbf{t}, \xi(\mathbf{t}, \mathbf{x})), \quad \xi(\mathbf{0}, \mathbf{x}) = \mathbf{x}$$

By Ambrosio-DiPerna-Lions (resp. Cauchy-Lipschitz) theory on ODEs, bounded variation (resp. Lipschitz) regularity of v in x is enough to preserve the volume (resp. the topology) of level sets.

The usual diffusion equation does not preserve the volume of level sets

The usual linear diffusion equation $\boxed{\partial_{\mathbf{t}}\varphi = \triangle\varphi}$ **cannot be written in form**

$$\boxed{\partial_{\mathbf{t}}\varphi + \nabla\cdot(\mathbf{v}\varphi) = \mathbf{0}, \quad \nabla\cdot\mathbf{v} = \mathbf{0}}$$

and, therefore, cannot preserve the volume of the level sets of φ**.**

A concept of "admissible" solutions

DEFINITION **We say that a pair $(\varphi, \mathbf{v})$ on $[0, T] \times \mathbb{T}^d$ is admissible if φ is transported by v**

$$\partial_{\mathbf{t}} \varphi + \nabla \cdot (\mathbf{v} \varphi) = \mathbf{0}$$

A concept of "admissible" solutions

DEFINITION **We say that a pair $(\varphi, \mathbf{v})$ on $[0, T] \times \mathbb{T}^d$ is admissible if φ is transported by v**

$$\partial_{\mathbf{t}} \varphi + \nabla \cdot (\mathbf{v} \varphi) = \mathbf{0}$$

and v is a divergence-free vector-field (with zero mean in x).

A concept of "admissible" solutions

DEFINITION **We say that a pair $(\varphi, \mathbf{v})$ on $[0, T] \times \mathbb{T}^d$ is admissible if φ is transported by v**

$$\partial_{\mathbf{t}} \varphi + \nabla \cdot (\mathbf{v} \varphi) = \mathbf{0}$$

and v is a divergence-free vector-field (with zero mean in x). When v has Sobolev regularity $W^{s,2}$ in x, $s \geq 1$ is enough to enforce the volume preservation of all level sets (DiPerna-Lions' theory),

A concept of "admissible" solutions

DEFINITION **We say that a pair $(\varphi, \mathbf{v})$ on $[0, T] \times \mathbb{T}^d$ is admissible if φ is transported by v**

$$\partial_{\mathbf{t}} \varphi + \nabla \cdot (\mathbf{v} \varphi) = \mathbf{0}$$

and v is a divergence-free vector-field (with zero mean in x). When v has Sobolev regularity $W^{s,2}$ in x, $s \geq 1$ is enough to enforce the volume preservation of all level sets (DiPerna-Lions' theory), but not their topology unless $s > d/2 + 1$.

A concept of "admissible" solutions

DEFINITION **We say that a pair $(\varphi, \mathbf{v})$ on $[0, T] \times \mathbb{T}^d$ is admissible if φ is transported by v**

$$\partial_{\mathbf{t}} \varphi + \nabla \cdot (\mathbf{v} \varphi) = \mathbf{0}$$

and v is a divergence-free vector-field (with zero mean in x). When v has Sobolev regularity $W^{s,2}$ in x, $s \geq 1$ is enough to enforce the volume preservation of all level sets (DiPerna-Lions' theory), but not their topology unless $s > d/2 + 1$. In the case $s = 0$, we will focus on for notational simplicity, we will speak only of "formal preservation".

Balance of energy for smooth admissible solutions

LEMMA 1 For any SMOOTH admissible pair $(\varphi, \mathbf{v})$, we have

$$\boxed{\frac{\mathbf{d}}{\mathbf{dt}}||\nabla\varphi||^2 = -\mathbf{2}((\mathbf{v}, \mathbf{Pg})) = ||\mathbf{v} - \mathbf{Pg}||^2 - ||\mathbf{v}||^2 - ||\mathbf{Pg}||^2}$$

$$\boxed{\mathbf{g} = -\nabla \cdot (\nabla\varphi \otimes \nabla\varphi)}$$

Here $||\cdot||$ and $((\cdot,\cdot))$ respectively denote the L^2 norm and inner product in space, while P denotes the L^2 Helmholtz projection onto divergence-free vector fields.

Balance of energy for smooth admissible solutions

LEMMA 1 For any SMOOTH admissible pair $(\varphi, \mathbf{v})$, we have

$$\frac{\mathbf{d}}{\mathbf{dt}}||\nabla\varphi||^2 = -\mathbf{2}((\mathbf{v}, \mathbf{Pg})) = ||\mathbf{v} - \mathbf{Pg}||^2 - ||\mathbf{v}||^2 - ||\mathbf{Pg}||^2$$

$$\mathbf{g} = -\nabla \cdot (\nabla\varphi \otimes \nabla\varphi)$$

Here $||\cdot||$ and $((\cdot,\cdot))$ respectively denote the L^2 norm and inner product in space, while P denotes the L^2 Helmholtz projection onto divergence-free vector fields. Observe that

$$\frac{\mathbf{d}}{\mathbf{dt}}||\nabla\varphi||^2 + ||\mathbf{v}||^2 + ||\mathbf{Pg}||^2 \leq \mathbf{0} \text{ if and only if } ||\mathbf{v} - \mathbf{Pg}||^2 = \mathbf{0}$$

Definition of our LSVP diffusion equation

We now introduce the LSVP diffusion equation:

$$\partial_{\mathbf{t}}\varphi + \nabla \cdot (\varphi \mathbf{v}) = \mathbf{0}, \quad \mathbf{v} = -\mathbf{P}\nabla \cdot (\nabla\varphi \otimes \nabla\varphi)$$

Definition of our LSVP diffusion equation

We now introduce the LSVP diffusion equation:

$$\partial_{\mathbf{t}}\varphi + \nabla \cdot (\varphi \mathbf{v}) = \mathbf{0}, \quad \mathbf{v} = -\mathbf{P}\nabla \cdot (\nabla\varphi \otimes \nabla\varphi)$$

obtained as a "gradient flow" by saturating the inequality in Lemma 1, à la De Giorgi, Ambrosio-Gigli-Savaré.

Definition of our LSVP diffusion equation

We now introduce the LSVP diffusion equation:

$$\partial_{\mathbf{t}}\varphi + \nabla \cdot (\varphi \mathbf{v}) = \mathbf{0}, \quad \mathbf{v} = -\mathbf{P}\nabla \cdot (\nabla\varphi \otimes \nabla\varphi)$$

obtained as a "gradient flow" by saturating the inequality in Lemma 1, à la De Giorgi, Ambrosio-Gigli-Savaré.
This evolution equation is very non-linear and the local existence of smooth solutions is not clear. However, Lemma 1 provides a variational characterization which is powerful enough to define a reasonable concept of generalized solutions, as shown later.

Relations with Physics and linear algebra

Physically speaking, the LSVP equation describes the Darcy flow (for s $= 0$, or Stokes flow, for s $= 1$) of an electrically charged incompressible fluid (v and φ being the velocity and the electric potential).

Relations with Physics and linear algebra

Physically speaking, the LSVP equation describes the Darcy flow (for s $= 0$, or Stokes flow, for s $= 1$) of an electrically charged incompressible fluid (v and φ being the velocity and the electric potential).

An analogous model is Moffat's magnetic relaxation, which can also be seen as Darcy MHD.

Relations with Physics and linear algebra

Physically speaking, the LSVP equation describes the Darcy flow (for s $=$ 0, or Stokes flow, for s $=$ 1) of an electrically charged incompressible fluid (v and φ being the velocity and the electric potential).

An analogous model is Moffat's magnetic relaxation, which can also be seen as Darcy MHD.

In this model a time-dependent 1-form $A_i(t,x)dx^i$ (the "magnetic potential") and a divergence-free vector field $v(t,x)$ (the velocity of the fluid) are said admissible if A is transported by v:

$$\partial_t A_i + \sum_j dA_{ij} v_j = 0, \quad dA_{ij} = \partial_j A_i - \partial_i A_j$$

Then the Moffat equation can be similarly obtained as the "gradient flow" of the Dirichlet integral $\int ||dA||^2 dx$.

Relations with Physics and linear algebra

With a suitable potential added to the Dirichlet integral and set on the unit ball instead of the torus, the LSVP equations have interesting special solutions which are linear in space

$$\nabla\varphi(\mathbf{t},\mathbf{x}) = \mathbf{M}(\mathbf{t})\mathbf{x}, \quad \mathbf{v}(\mathbf{t},\mathbf{x}) = \mathbf{B}(\mathbf{t})\mathbf{x}, \quad \mathbf{M} = \mathbf{M}^{\mathbf{T}}, \quad \mathbf{B} = -\mathbf{B}^{\mathbf{T}}$$

Relations with Physics and linear algebra

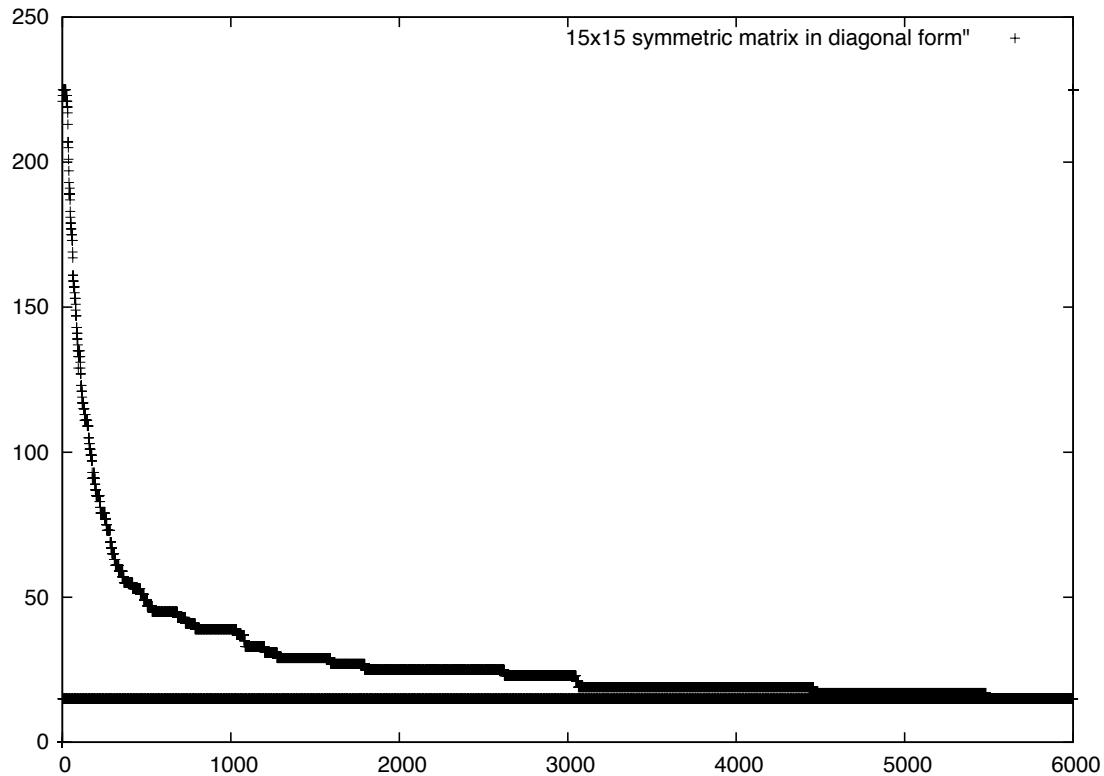
With a suitable potential added to the Dirichlet integral and set on the unit ball instead of the torus, the LSVP equations have interesting special solutions which are linear in space

$$\nabla\varphi(t,x) = M(t)x, \quad v(t,x) = B(t)x, \quad M = M^T, \quad B = -B^T$$

Then, we recover the Brockett diagonalizing gradient flow for symmetric matrices (recently studied by V. Bach and J.-B. Bru, as told us by M. Salmhofer):

$$\frac{dM}{dt} = [B,M], \quad B = [M,Q]$$

LSVP DIAGONALIZATION OF SYMM. MATRICES



For a given 15x15 symmetric matrix (with random coefficients) evolving according to the LSVP equation (with external force), the number of off-diagonal coefficients below 0.001 tends to 15 after time 60 (calculation performed with 6000 time steps).

LEMMA 2 **For** $\mathbf{g} = -\nabla \cdot (\nabla\varphi \otimes \nabla\varphi)$

$$||\mathbf{Pg}||^2 = \sup_{\mathbf{r}\geq \mathbf{0}} \ \mathbf{K_r}(\varphi) - \mathbf{r}||\nabla\varphi||^2$$

where $\mathbf{K_r}(\varphi)$ **is convex and defined, for** $\mathbf{r} \geq \mathbf{0}$**, in** $[\mathbf{0}, +\infty]$ **as**

$$\sup_{\mathrm{Eigen}(\partial_i z_k + \partial_k z_i + r\delta_{ik}) \geq 0,\ \partial_k z_k = 0} \int \partial_i\varphi \partial_k\varphi (\partial_i z_k + \partial_k z_i + r\delta_{ik})\ dx - ||z||^2.$$

LEMMA 2 **For** $\mathbf{g} = -\nabla \cdot (\nabla\varphi \otimes \nabla\varphi)$

$$\boxed{||\mathbf{Pg}||^2 = \sup_{\mathbf{r}\geq \mathbf{0}} \ \mathbf{K_r}(\varphi) - \mathbf{r}||\nabla\varphi||^2}$$

where $\mathbf{K_r}(\varphi)$ **is convex and defined, for** $\mathbf{r} \geq \mathbf{0}$**, in** $[\mathbf{0}, +\infty]$ **as**

$$\sup_{\mathrm{Eigen}(\partial_i z_k + \partial_k z_i + r\delta_{ik})\geq 0,\ \partial_k z_k = 0} \int \partial_i\varphi\partial_k\varphi(\partial_i z_k + \partial_k z_i + r\delta_{ik})\ dx - ||z||^2.$$

Proof:

$$||Pg||^2 = \sup_{\nabla\cdot z = 0} \ 2((Pg, z)) - ||z||^2$$

$$= \sup_{\nabla\cdot z = 0} \int \partial_i\varphi\partial_k\varphi(\partial_i z_k + \partial_k z_i)\ dx - ||z||^2$$

LEMMA 2 **For** $\mathbf{g} = -\nabla \cdot (\nabla\varphi \otimes \nabla\varphi)$

$$\boxed{||\mathbf{Pg}||^2 = \sup_{\mathbf{r}\geq \mathbf{0}} \ \mathbf{K_r}(\varphi) - \mathbf{r}||\nabla\varphi||^2}$$

where $\mathbf{K_r}(\varphi)$ **is convex and defined, for** $\mathbf{r} \geq \mathbf{0}$**, in** $[\mathbf{0}, +\infty]$ **as**

$$\sup_{\text{Eigen}(\partial_i z_k + \partial_k z_i + r\delta_{ik}) \geq 0,\ \partial_k z_k = 0} \int \partial_i\varphi \partial_k\varphi (\partial_i z_k + \partial_k z_i + r\delta_{ik})\ dx - ||z||^2.$$

Proof:

$$||Pg||^2 = \sup_{\nabla\cdot z = 0} \ 2((Pg, z)) - ||z||^2$$

$$= \sup_{\nabla\cdot z = 0} \int \partial_i\varphi \partial_k\varphi (\partial_i z_k + \partial_k z_i)\ dx - ||z||^2$$

$$= \sup_{r\geq 0} \ \sup_{\text{Eigen}(\partial_i z_k + \partial_k z_i + r\delta_{ik}) \geq 0,\ \partial_k z_k = 0} \int \partial_i\varphi \partial_k\varphi (\partial_i z_k + \partial_k z_i)\ dx - ||z||^2$$

Dissipative solutions

DEFINITION **We say that an admissible pair $(\varphi, \mathbf{v})$ is a dissipative solution to the diffusion equation with initial condition $\varphi(0) = \varphi_{\mathbf{ini}}$ if for every nonnegative function $t \to r(t) \geq 0$**

$$" \quad (\frac{d}{dt} - r)||\nabla\varphi||^2 + ||\mathbf{v}||^2 + K_r(\varphi) \leq 0 \quad "$$

holds true in integral form from 0 to t, i.e., for $R(t) = \int_0^t r(\tau)d\tau$,

$$||\nabla\varphi(t)||^2 + \int_0^t e^{R(t)-R(\tau)}[||\mathbf{v}(\tau)||^2 + K_{r(\tau)}(\varphi(\tau))]d\tau \ \leq \ ||\nabla\varphi_{\mathbf{ini}}||^2 e^{R(t)}$$

(which is a convex inequality).

SOME RESULTS

THEOREM

1) The energy inequality is convex and, therefore, weakly stable.

SOME RESULTS

THEOREM

1) The energy inequality is convex and, therefore, weakly stable.

2) The concept of admissible solutions is weakly closed.

SOME RESULTS

THEOREM

1) The energy inequality is convex and, therefore, weakly stable.
2) The concept of admissible solutions is weakly closed.
3) For every initial condition $\varphi_{ini} \in W^{1,2}$, there is always at least a global dissipative solution.

SOME RESULTS

THEOREM

1) The energy inequality is convex and, therefore, weakly stable.
2) The concept of admissible solutions is weakly closed.
3) For every initial condition $\varphi_{ini} \in W^{1,2}$, there is always at least a global dissipative solution.
4) For every initial condition $\varphi_{ini} \in W^{1,2}$ the set of dissipative solutions has a unique element, whenever it has a smooth one.

SOME RESULTS

THEOREM

1) The energy inequality is convex and, therefore, weakly stable.
2) The concept of admissible solutions is weakly closed.
3) For every initial condition $\varphi_{ini} \in W^{1,2}$, there is always at least a global dissipative solution.
4) For every initial condition $\varphi_{ini} \in W^{1,2}$ the set of dissipative solutions has a unique element, whenever it has a smooth one.
Of course, many problems are pending: smoothness of solutions, actual preservation of the topology (not guaranteed by dissipative solutions), possible disruptions and reconnections of level sets, convergence to stationary solutions of the Euler equations in 2D...

RELATED WORKS

1. L. Ambrosio, N. Gigli, G. Savaré's book on gradient flows and recent work on the heat equation on general metric spaces.
2. V.I. Arnold, B. Khesin, Topological methods in hydrodynamics, Springer 1998.
3. Y. Brenier, Topology-preserving diffusion of divergence-free vector fields and magnetic relaxation, arXiv:1304.4562, to appear in CMP.
4. R. W. Brockett, Linear Algebra and its applications, 146 (1991) 79-91.
5. N. Ghoussoub, Self-dual Partial Differential Systems and Their Variational Principles, Springer 2008.
6. P.-L. Lions' book on fluid mechanics (tome I).
7. H.K. Moffatt, Relaxation under topological constraints, ITP, Santa-Barbara 1991.

RELATED WORKS

1. L. Ambrosio, N. Gigli, G. Savaré's book on gradient flows and recent work on the heat equation on general metric spaces.
2. V.I. Arnold, B. Khesin, Topological methods in hydrodynamics, Springer 1998.
3. Y. Brenier, Topology-preserving diffusion of divergence-free vector fields and magnetic relaxation, arXiv:1304.4562, to appear in CMP.
4. R. W. Brockett, Linear Algebra and its applications, 146 (1991) 79-91.
5. N. Ghoussoub, Self-dual Partial Differential Systems and Their Variational Principles, Springer 2008.
6. P.-L. Lions' book on fluid mechanics (tome I).
7. H.K. Moffatt, Relaxation under topological constraints, ITP, Santa-Barbara 1991.