Imaging of flow in porous media - from optimal transport to prediction

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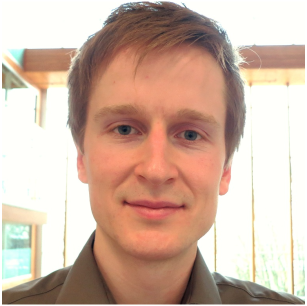
With

Rowan Cocket



Lars Ruthotto

Jenn Fohring

Outline

Prediction is very difficult, especially about the future.

Niels Bohr

Outline

- Multiphysics imaging
- The mathematical problem
- Discretization
- Solution through Variable Projection
- Summary and future work

Flow in porous media

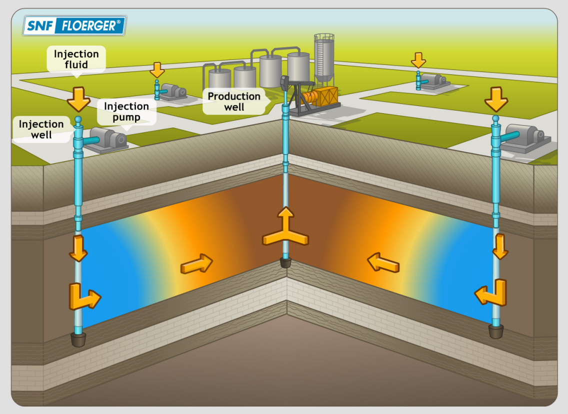
Flow in porous media is used for

- Enhanced Oil Recovery
- CO_2 sequestration monitoring
- Salt water intrusion monitoring

Enhanced Oil Recovery

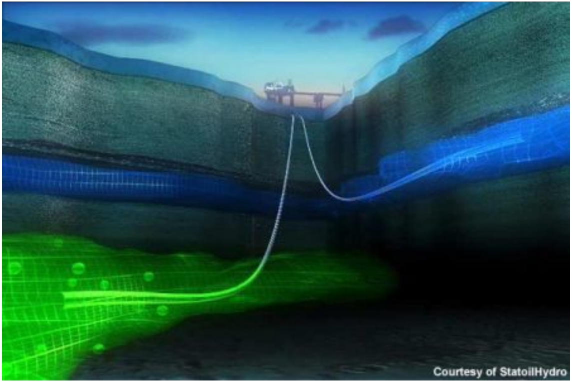
Inject CO_2 to push oil out

Goal: image and control the flow



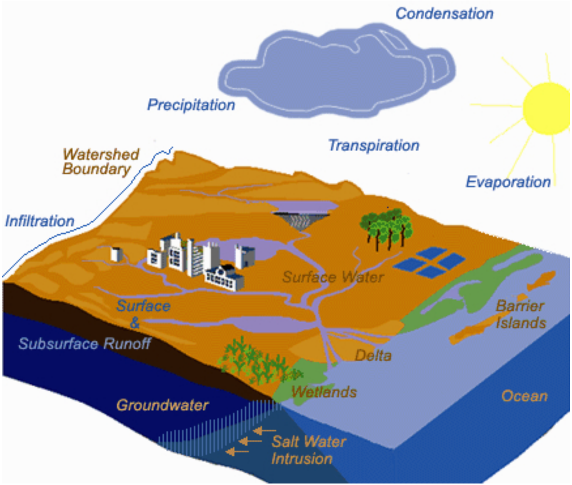
CO_2 Sequestration monitoring

Is the CO_2 staying in the ground?
Where does it flow to?



Salt water intrusion monitoring

Is salt water polluting fresh water aquifer?



Flow in porous media

Governing equations (IMPES formulation)

$$
\begin{aligned}
\nabla \cdot \vec{\mathbf{u}} &= q && \text{IMP} \\
\vec{\mathbf{u}} &= \lambda_s(s)\kappa\nabla p && \\
s_t + \nabla \cdot \left(\vec{\mathbf{u}}\,\lambda(s)\right) &= 0 && \text{ES}
\end{aligned}
$$

- Given s_0 and parameters possible to solve for p and $s(t)$
- In realistic situations κ, λ and s_0 are known to very low accuracy (or not at all)
- Difficult to predict the flow

Flow in porous media

Prediction is very difficult
Long term prediction impossible

Improving prediction

- Drill
- History match well data

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.

- Use imaging to "see" the fluids

Imaging flow

In general, consider the dynamical system

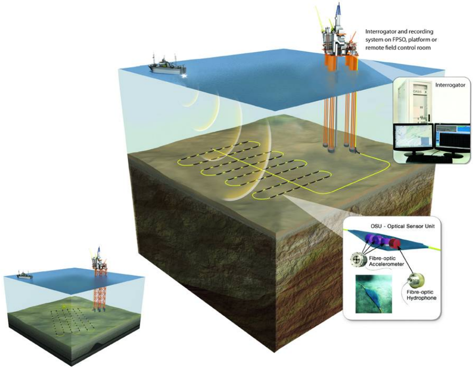
$$\dot{s} = f(s, u) \quad s(0) = s_0$$

- Dynamical system with uncertain inputs
- Let the dynamics run for a short time and use data to update parameters
- Improve flow model
- Data assimilation

Imaging flow

- Use time laps imaging for fluid flow
- Fluids change the physical properties

Goal: Combine imaging and dynamics to better predict the flow



Imaging fluids and flow

Electromagnetic methods

$$\nabla \times \mu^{-1} \nabla \times \vec{\mathbf{e}} + i\omega\sigma(s)\vec{\mathbf{e}} = i\omega\vec{\mathbf{q}}$$
$$d = Q\vec{\mathbf{e}} = Q\mathcal{F}(\sigma)$$

$\vec{\mathbf{e}}$ - electric field σ - conductivity

Seismic methods

$$\Delta u + \omega^2\gamma(s)u = q$$
$$d = Qu = Q\mathcal{F}(\gamma)$$

u - pressure field γ - seismic velocity

In general: $\mathcal{F}(m) + \epsilon = d$

Model Flow Problem - Tracer flow

Flow equations

$$
\begin{aligned}
\nabla \cdot \vec{\mathbf{u}} &= q \\
\vec{\mathbf{u}} &= \kappa(x)\nabla p \\
s_t + \nabla \cdot (\vec{\mathbf{u}}\ s) &= 0
\end{aligned}
$$

s - saturation

p - pressure

κ - hydraulic conductivity tensor

Model Imaging - Borehole tomography

Place sources and receivers in boreholes/surface and measure seismic/electric fields

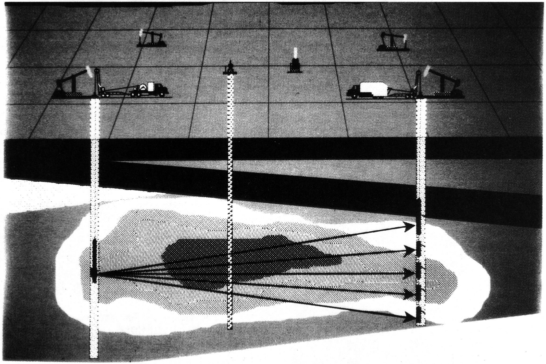


Figure 1. Schematic illustration of tomographic data acquisition.

Assumptions

- Flow
  $s_t + \nabla \cdot (\vec{\mathbf{u}}(\kappa, p)s) = 0 \quad s(0, x) = s_0$
- The imaging problem is linear w.r.t s
  Tomography
  $d(t) = As(t) + \epsilon$

Goals

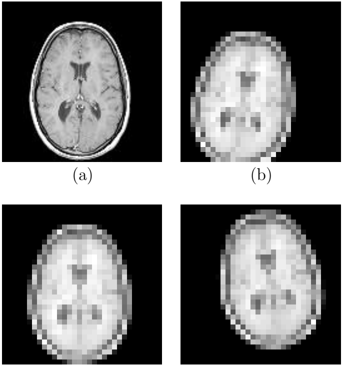
Prediction and control

$$s_t + \nabla \cdot (\vec{\mathbf{u}}(\kappa, p)s) = 0 \quad s(0, x) = s_0$$
$$As(t) + \epsilon = d$$

- No need for the pressure!
- Recover the velocity $\vec{\mathbf{u}}$ and the saturation s

Similarity to super resolution

Super Resolution - Use a number of low-res images to obtain a single high-res image



$$A\,I(u)s + \epsilon = d$$

Similarity to super resolution

Super Resolution - Use a number of low-res images to obtain a single high-res image

- Solve for s ans $\vec{u}$
- Similar to the problem of super resolution [Elad & Furer, 90, Chung, H & Nagy 06, Borzi & Kunisch 07]
- Main differences - More complex dynamics and observation operators
- Similar mathematical structure

Solution through optimization

$$\begin{array}{ll} \min\limits_{s,\vec{\mathbf{u}}} & \mathcal{J}(s_0, \vec{\mathbf{u}}) \\ \text{s.t.} & s_t + \nabla \cdot (\vec{\mathbf{u}} s) = 0 \quad s(0,x) = s_0 \end{array}$$

- Similar to the optimal control approach to OMT of Benamou & Brenier
- But there are major differences

Solution through optimization

$$\begin{aligned} \min_{s,\vec{\mathbf{u}}} \quad & \mathcal{J}(s_0, \vec{\mathbf{u}}) = \sum_j \|As(t_j) - d_j\|^2 + \alpha_s R_s(s) + \alpha_u R_u(\vec{\mathbf{u}}) \\ \text{s.t.} \quad & s_t + \nabla \cdot (\vec{\mathbf{u}} s) = 0 \quad s(0,x) = s_0 \end{aligned}$$

Solution through optimization

$$\min_{s,\vec{\mathbf{u}}} \quad \mathcal{J}(s_0, \vec{\mathbf{u}}) = \sum_j \|As(t_j) - d_j\|^2 + \alpha_s R_s(s) + \alpha_u R_u(\vec{\mathbf{u}})$$

$$\text{s.t.} \quad s_t + \nabla \cdot (\vec{\mathbf{u}} s) = 0 \quad s(0, x) = s_0$$

- Optimal* mass transport - **optimality criteria based on data**
- OMT does not have a unique solution and require regularization
- Choice of regularization- **motivated by the physics of the problem**

Solution through optimization

Continuous problem $\min \mathcal{J}(u)$

Discretize then optimize	**Optimize then discretize**
Discretize u and $\mathcal{J}$ compute $\mathbf{g}(\mathbf{u}) = \vec{\nabla}_{\mathbf{u}} J(\mathbf{u})$ Solve the discrete problem	Compute $g(u) = \vec{\nabla}_u \mathcal{J}(u) = 0$ Discretize $g_h(\mathbf{u}) = 0$ Solve the discrete PDE

Solution through optimization

Continuous problem $\min \mathcal{J}(u)$

Discretize then optimize	Optimize then discretize
Discretize u and $\mathcal{J}$	Compute $g(u) = \vec{\nabla}_u \mathcal{J}(u) = 0$
compute $\mathbf{g}(\mathbf{u}) = \vec{\nabla}_{\mathbf{u}} J(\mathbf{u})$	Discretize $g_h(\mathbf{u}) = 0$
Solve the discrete problem	Solve the discrete PDE

- In general $g_h(\mathbf{u}) \neq \mathbf{g}(\mathbf{u})$
- $g_h(\mathbf{u})$ is not a gradient of **any** discrete function
- No guaranteed descent
- Convergence only when h is "small enough"

Solution through optimization

Our framework: **Discretize and optimize**

- ☺ Gradient of the discrete function can be calculated exactly (linear algebra vs calculus)
- ☺ Best optimization algorithms can be used
  - Gradient flow = steepest decent (ssssslllllooooowwww)
  - Variations of Newton's method
  - Multilevel Newton methods
- ☹ How to discretize the hyperbolic PDE?

Discretization of the PDE

$$s_t + \nabla \cdot (\vec{\mathbf{u}}\, s) = 0 \quad s(0, x) = s_0$$

Some things to consider

- $\vec{\mathbf{u}}$ unknown - CFL condition unknown
- Unconditionally stable methods
- Upwinding - non-differentiable!
- Most high resolution methods (ENO, WENO,) are highly nonlinear and non-differentiable
- Keeping discontinuities not relevant(?)

Discretization of the PDE

Explicit methods

- Careful control over time stepping
- Differentiability - no flux limiters

Implicit methods

- No stability issues
- Invert linear systems

Semi-Lagrangian methods

- No stability issues
- Can be designed to be differentiable

Example for difficulty - Explicit Methods

Test Equation: $s_t - us_x = 0$

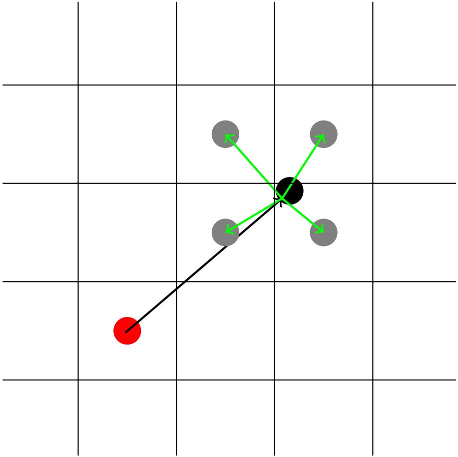
- Upwind

$$s_{k+1} = s_k + \frac{\Delta t}{\Delta x}\left(\overbrace{\max(u,0)D^+ + \min(u,0)D^-}^{\text{non differentiable}}\right) s_k$$

- Lax - Friedrichs

$$s_{k+1} = A_v s_k + \frac{\Delta t}{2\Delta x}\text{diag}(u)D^c s_k$$

Discretization - Particle in Cell



PIC Discretization

Can be written as

$$s_{k+1} - \mathcal{I}(\mathbf{u})s_k = 0 \quad s(0,x) = s_0$$

- Exact conservation
- Unconditionally stable
- Can be made differentiable [H. Modersitzki, 06]

- Low accuracy
- Low diffusion

The discrete optimization problem

$$\begin{aligned} \min_{s,\vec{\mathbf{u}}} \quad & \frac{1}{2}\sum_j \|A\, s(t_j) - d(t_j)\|^2 + \alpha_s R_s(s_0) + \alpha_u R_u(\mathbf{u}) \\ \text{s.t.} \quad & s_{k+1} - \mathcal{I}(\mathbf{u}) s_k = 0 \quad s(0,x) = s_0 \end{aligned}$$

To complete need to choose regularization scheme

Choosing regularization for s_0

- Problem highly ill-posed, L_1 & TV not appropriate choice [Schwarzbach & H 12, Ascher, van Den Doel & H. 12]

Choice of regularization for s_0

- Smoothness

$$R_s(s_0) = \frac{1}{2}\int_\Omega \|\vec{\nabla} s_0\|^2 dV$$

- Weighted smoothness

$$R_s(s_0) = \frac{1}{2}\int_\Omega w(x)\|\vec{\nabla} s_0\|^2 dV$$

w - weighted support

Choosing regularization for $\vec{\mathbf{u}}$

$\vec{\mathbf{u}}$ - vector quantity

Recall that

- $\nabla \cdot \vec{\mathbf{u}} = 0 \quad \text{AE}$
- $\vec{\mathbf{u}}$ can have discontinuous tangential components

$$|\nabla \times \vec{\mathbf{u}}| \quad \text{jumpy}$$

Set

$$R(\vec{\mathbf{u}}) = \int_{\Omega} \frac{\alpha_1}{2} \|\nabla \cdot \vec{\mathbf{u}}\|^2 + \alpha_2 |\nabla \times \vec{\mathbf{u}}|_1 \, dV$$

The discrete optimization problem

$$\min_{s,\mathbf{u}} \quad \frac{1}{2}\sum_j \|As(t_j) - d(t_j)\|^2 + \alpha_s R_s(s_0) + \alpha_u R_u(\mathbf{u})$$

$$\text{s.t.} \quad s_{k+1} - \mathcal{I}(\mathbf{u}) s_k = 0 \quad s(0,x) = s_0$$

The problem is linear in s nonlinear in $\mathbf{u}$
Use Variable Projection (VarPro) [Golub Pereyra (73,02)]

Solution through Variable Projection

Eliminate Constraint $s = F(\mathbf{u})^{-1} I_0 s_0$ where

$$F(\mathbf{u}) = \begin{pmatrix} I & & & \\ -\mathcal{I}(\mathbf{u}) & I & & \\ & \ddots & \ddots & \\ & & -\mathcal{I}(\mathbf{u}) & I \end{pmatrix} \quad I_0 = \begin{pmatrix} -\mathcal{I}(\mathbf{u}) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Unconstrained problem

$$\min_{s_0, \mathbf{u}} \ \frac{1}{2} \| A F(\mathbf{u})^{-1} I_0 s_0 - d \|^2 + \alpha_s R_s(s_0) + \alpha_u R_u(\mathbf{u})$$

Solution through Variable Projection

Two step iteration [Chung, Nagy & H (06), Chung Thesis (08)]

- Minimize wrt s_0

$$\widehat{s}_0^{(k)} = \left(I_0^\top F^{-\top} A^\top A F^{-1} I_0 + \alpha_s \nabla^2 R_s\right)^{-1} F^{-\top} A^\top d$$

- Fix $s_0 = \widehat{s}_0^{(k)}$ and minimize over $\mathbf{u}$

$$\min_{\mathbf{u}} \ \frac{1}{2}\|AF(\mathbf{u})^{-1} I_0 \widehat{s}_0^{(k)} - d\|^2 + \alpha_u R_u(\mathbf{u})$$

Advantages

- Decoupling the inverse problems
- Easy to choose regularization parameters

Solution through Variable Projection

- No need to form matrices
- Use GCV for regularization parameter for s_0
- Lagged diffusivity for the $|\nabla \times \mathbf{u}|_1$ regularization [Vogel (96)]
- Solution of the problem for $\mathbf{u}$ need not be exact

Example - Imaging CO_2 Flow

Experimental Setting: Borehole Experiment

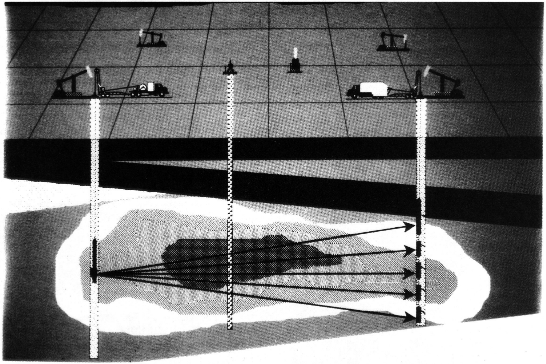
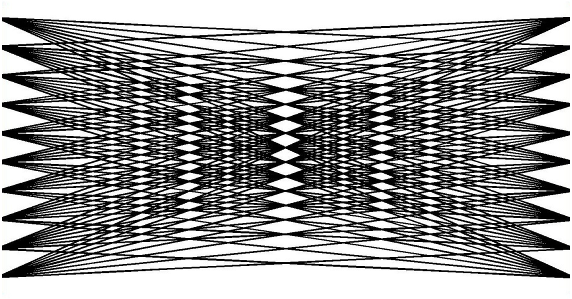


Figure 1. Schematic illustration of tomographic data acquisition.

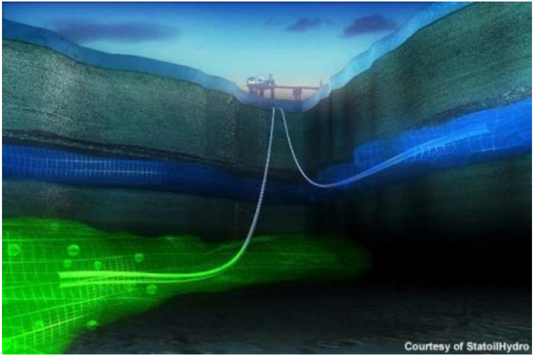
Example - Imaging CO_2 Flow

- Assume 625 rays (data points)
- 20 times observed
- Prediction after
- Velocity field obtained by solving the pressure equation with highly discontinuous coefficients



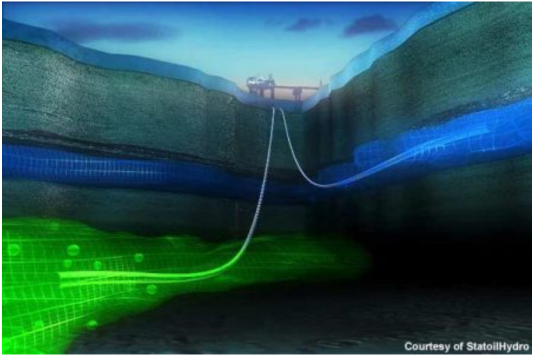
Example - Imaging CO_2 Flow

Flow simulation



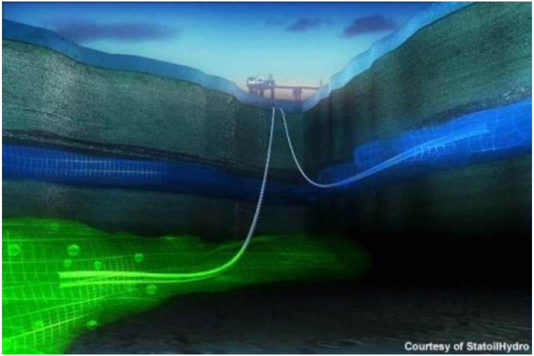
Example - Imaging CO_2 Flow

Observed Data



Recovered and predicted flow

Flow simulation



Comments

Reconstruction

- Excellent reconstruction of initial saturation
- Reasonable recovery of flow field

Prediction

- Short term predictions - excellent
- Long term prediction - fail
- No information on the velocity in regions where there is no flow

Summary and prediction

Summary

- Combine flow in porous media and imaging
- Basic framework - super resolution
- Requires special regularization
- VarPro for the solution

Prediction

- Algorithm speedup
- Use joint inversion criteria for unknown petrology
- Experimental design