Calculus of variations under convexity-like constraints

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Joint work with Quentin Mérigot

Motivations

($\mathcal{H}$ = space of convex functions)

Dual formulation of OT: $\max_{\phi \in \mathcal{H}} \int \phi(x) \, \mathrm{d}\, \mu(x) + \int \phi^*(y) \, \mathrm{d}\, \nu(y)$

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numerical applications in 1D [Blanchet, Calvez, Carillo '08]

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Geometric problems: e.g. Meissner conjecture

Geometric difficulty $(d \geq 2)$

Theorem: Any PL convex function on the regular grid (A) on $[0,1]^2$ satisfies in a suitable weak sense the inequality $\frac{\partial^2 \phi}{\partial x \partial y} \geq 0$.

[Choné-Le Meur '99]

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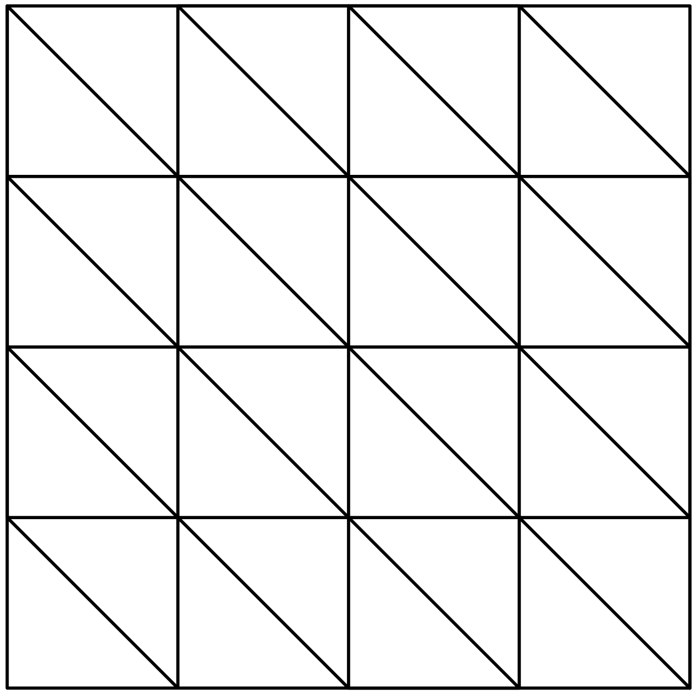

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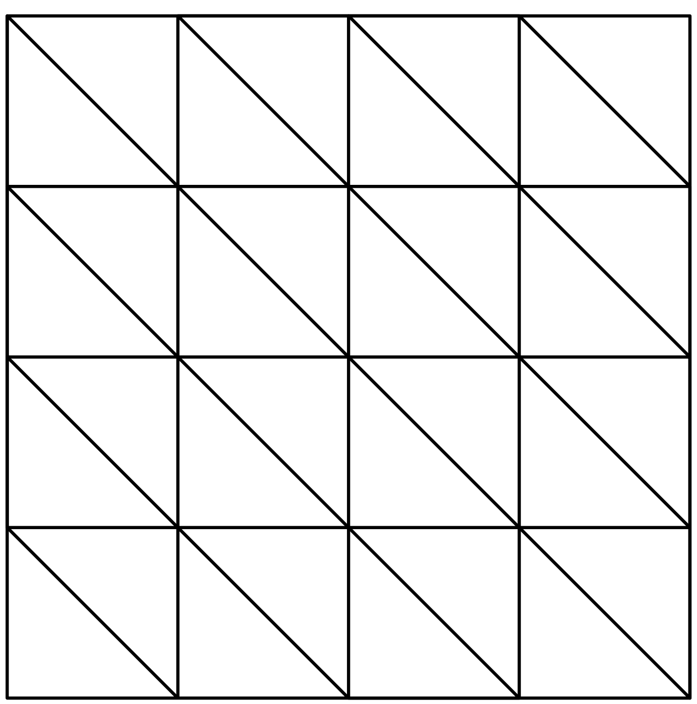

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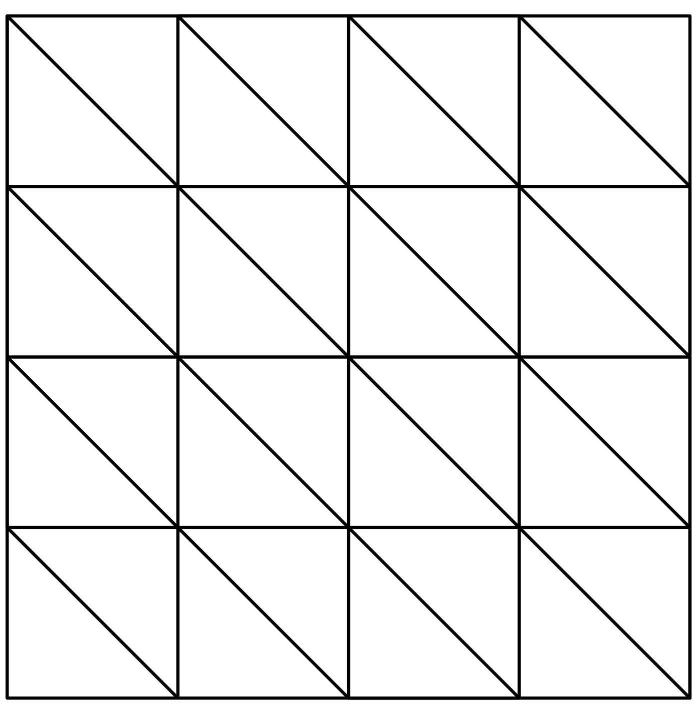

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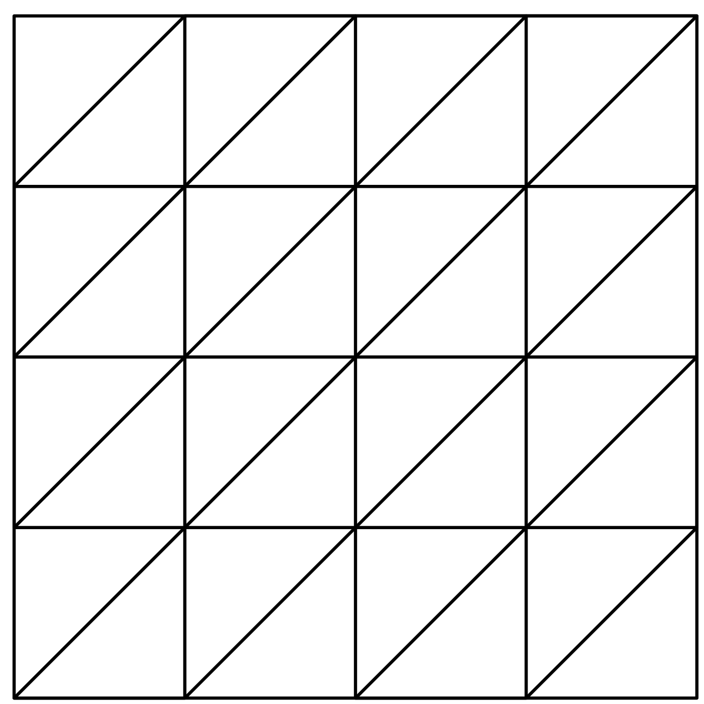

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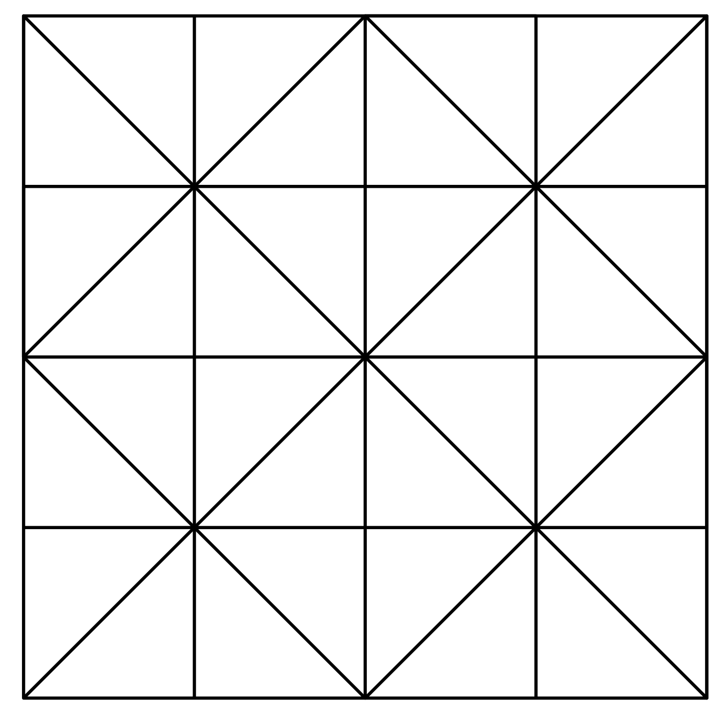

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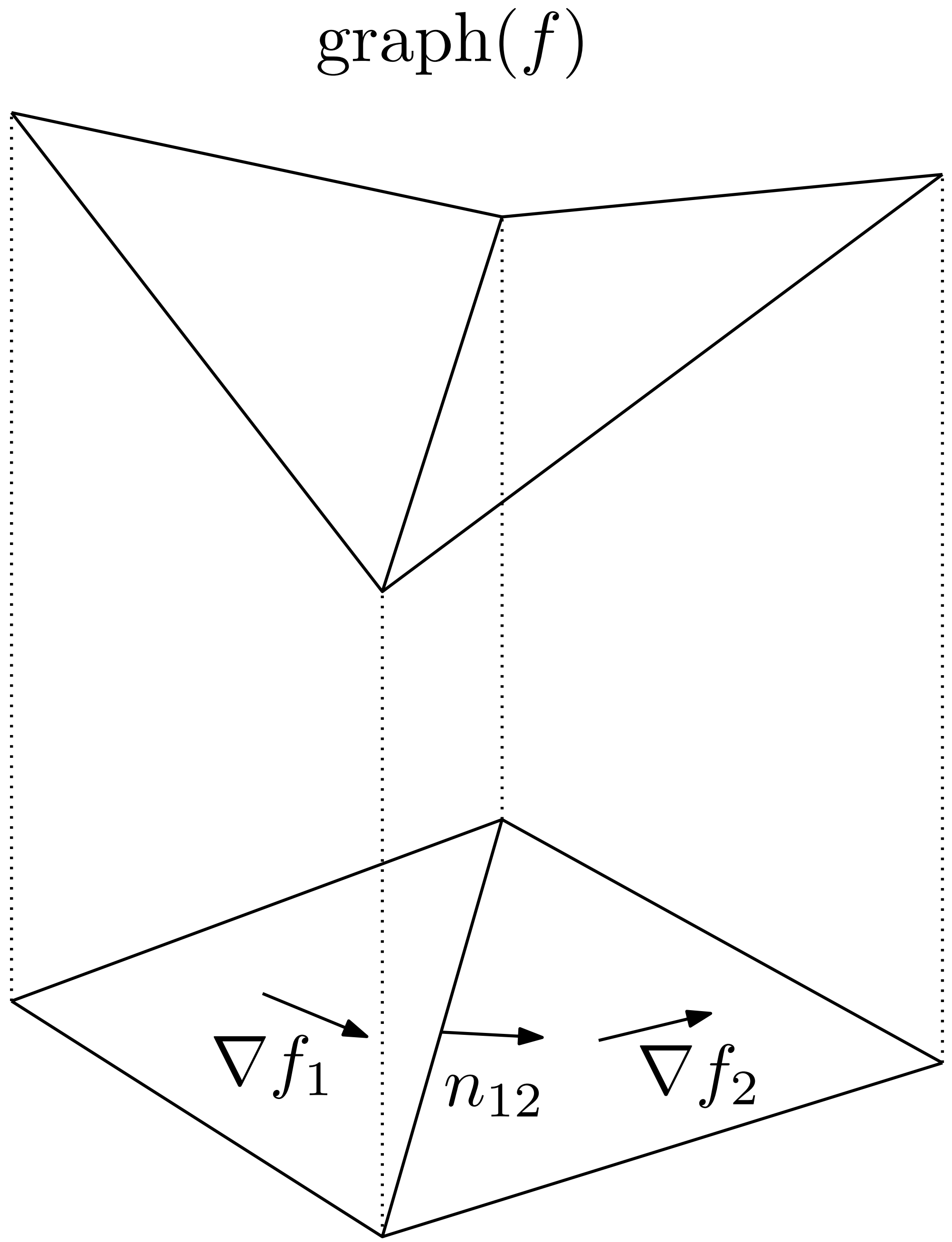

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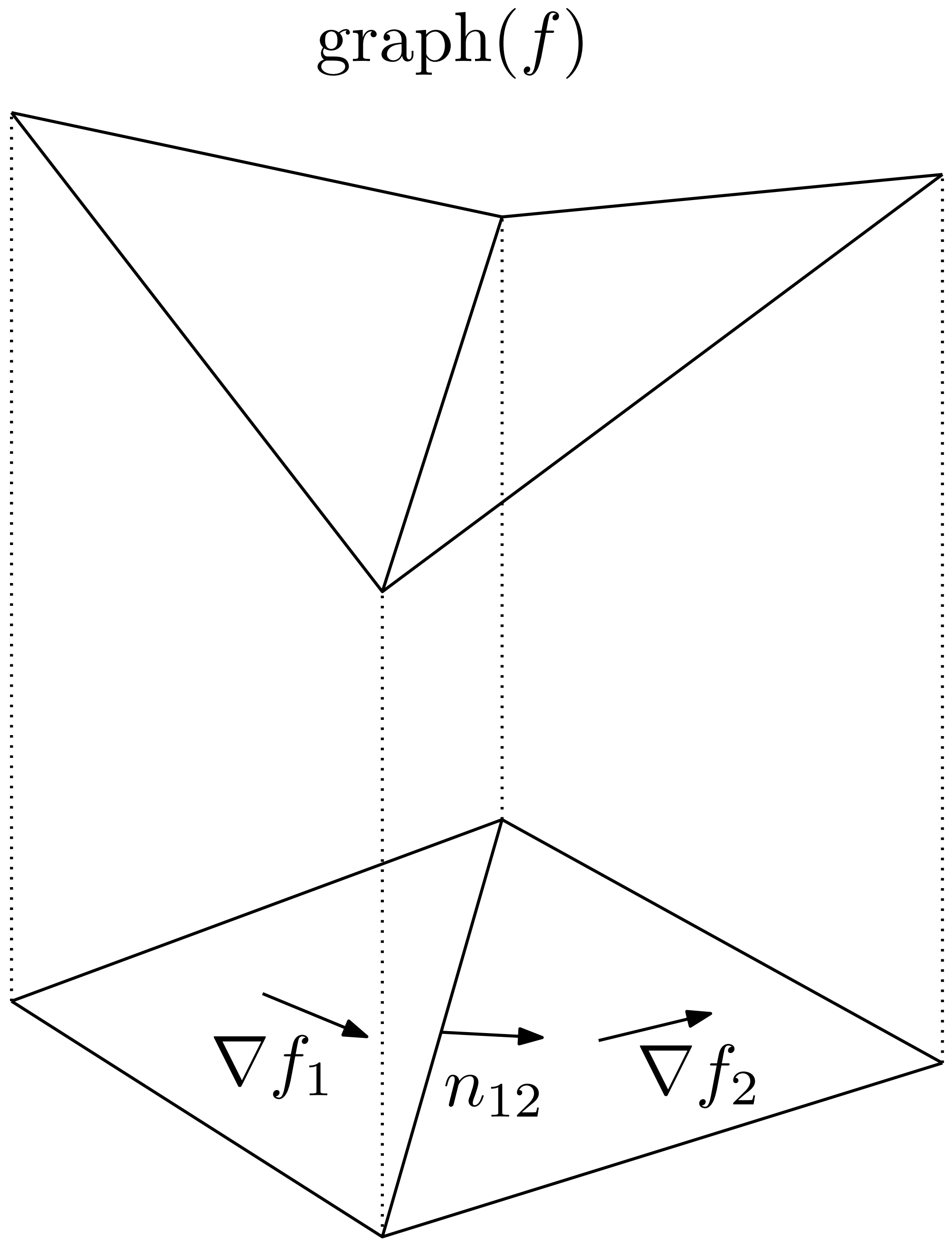

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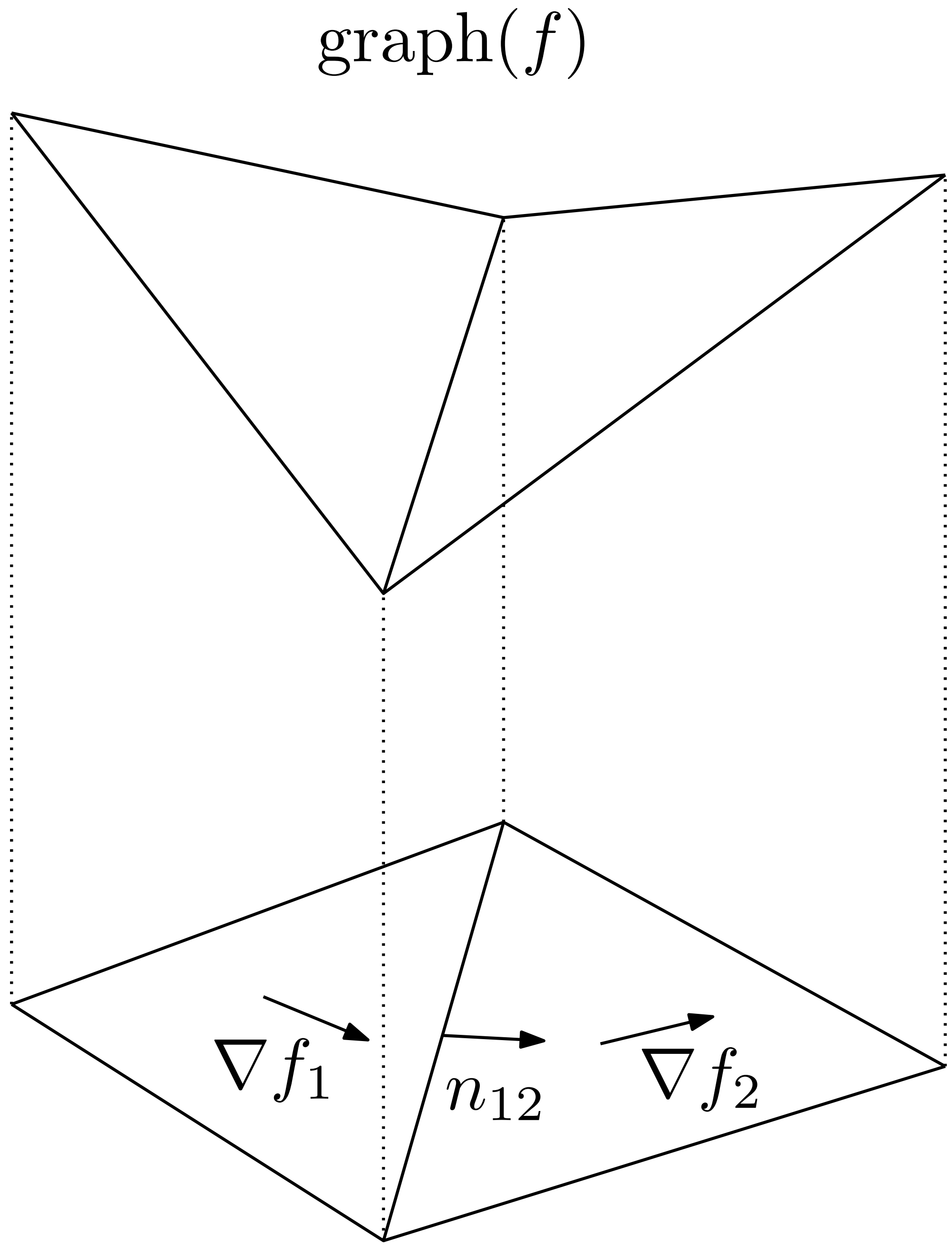

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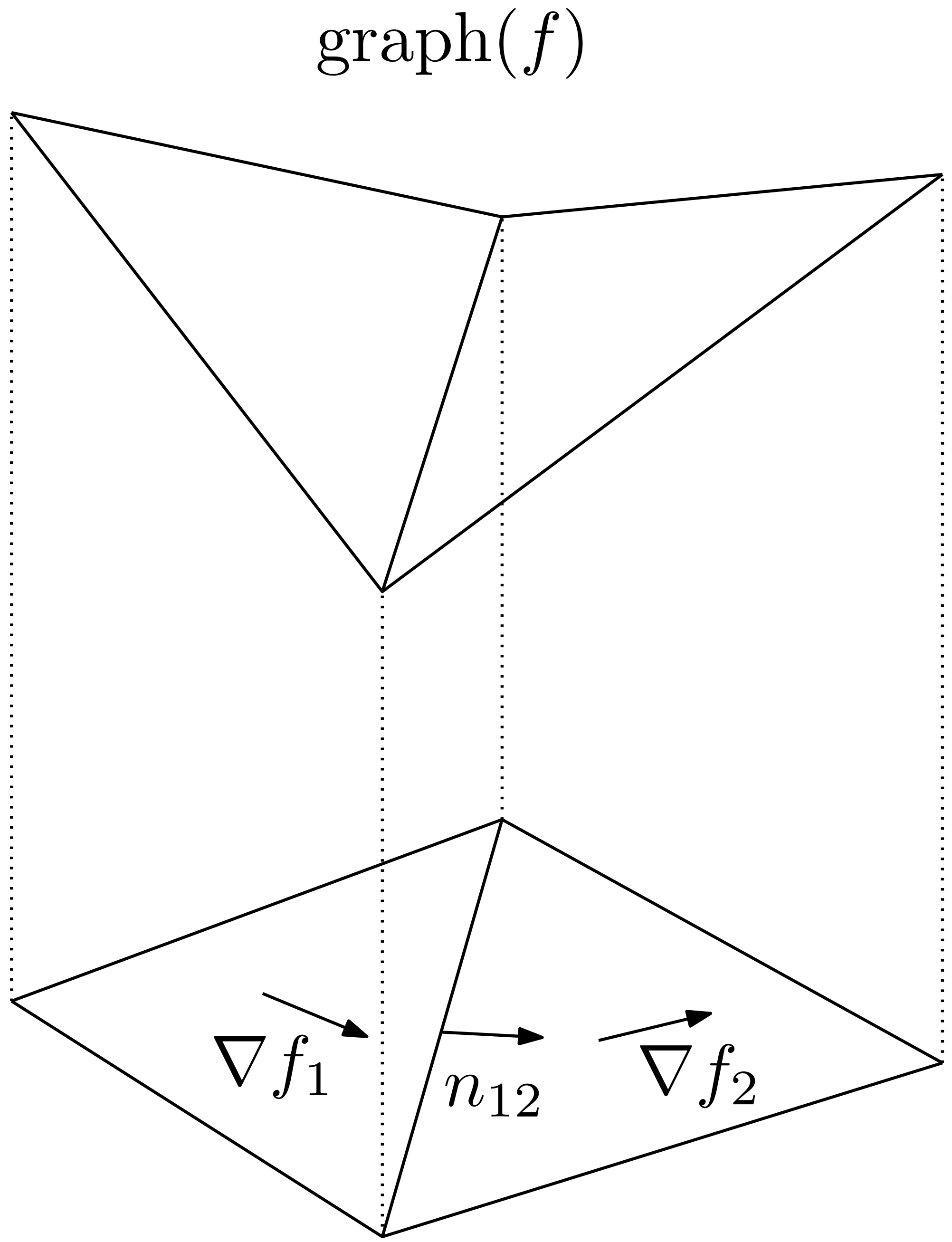

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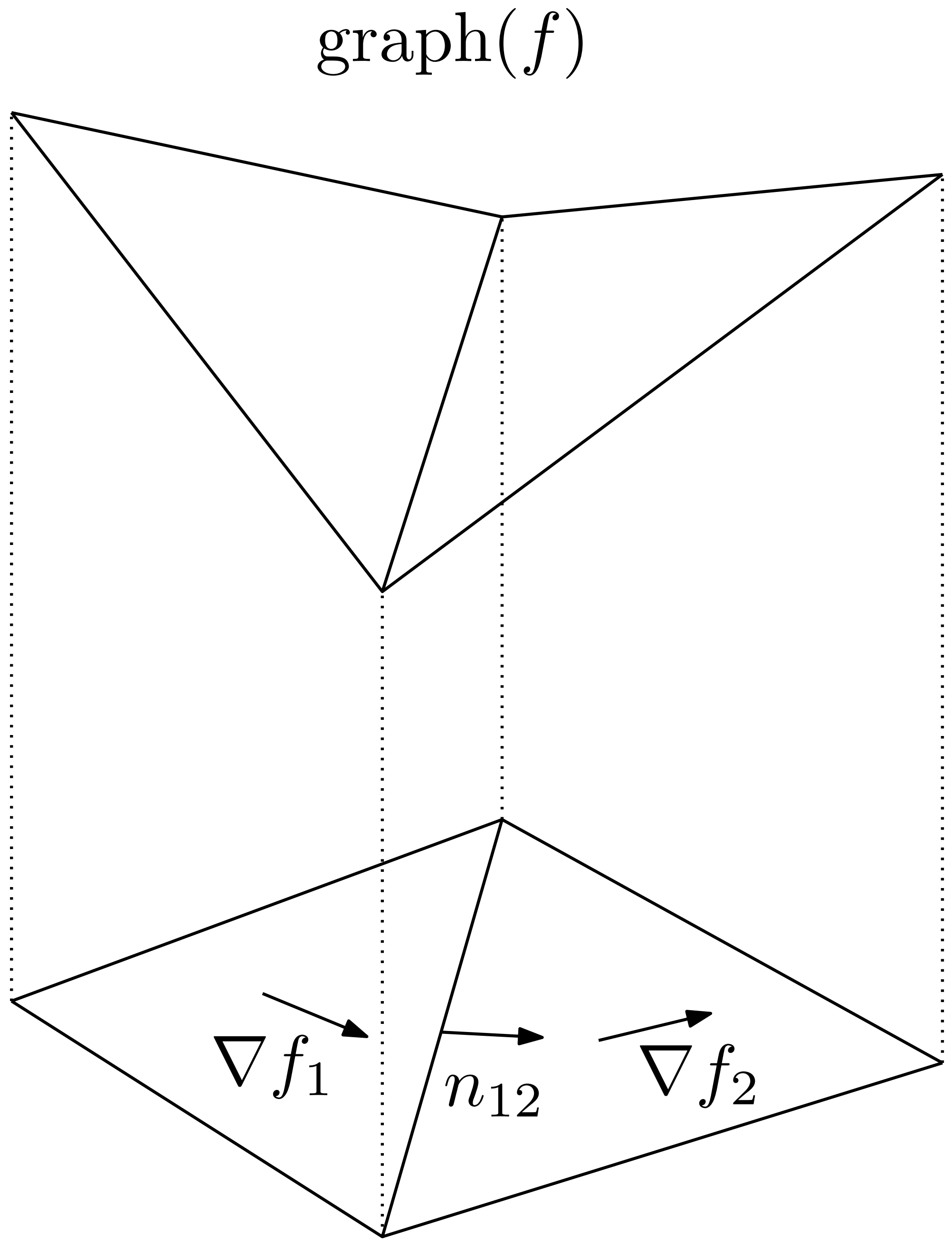

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4. for the grid (A), $\frac{\partial^2 f}{\partial x \partial y} \geq 0$.

Geometric difficulty: a numerical example

$f(x, y) := \max(0, x + y - 1)$ on $X = [0, 1]^2$

$g_\delta := \mathrm{L}^2$ projection of f on the space of PL functions on the grid (A)

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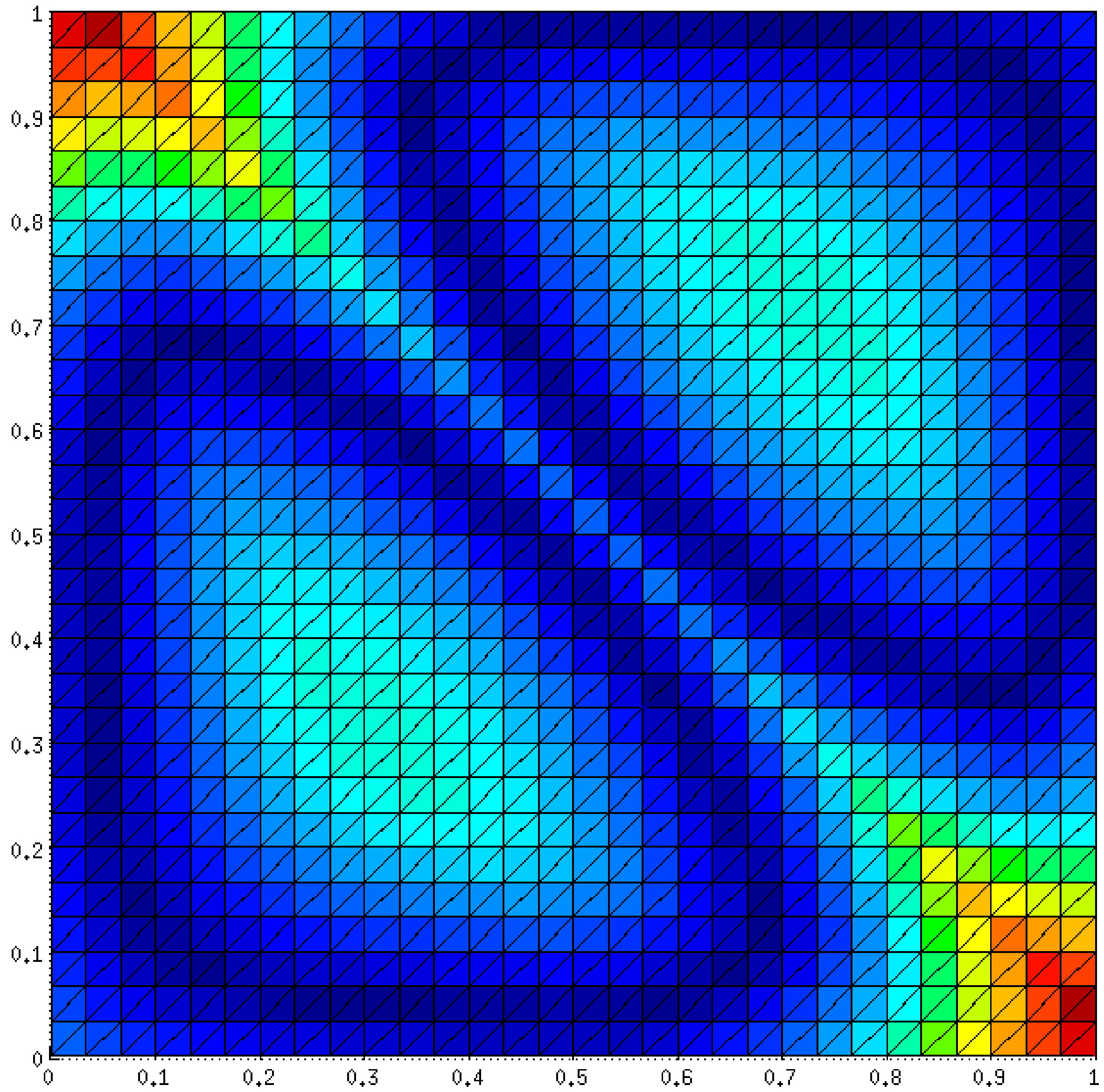

$|f(.) - g_\delta(.)|$ for $\delta = 1/30$

(red $\simeq$ 0.2)

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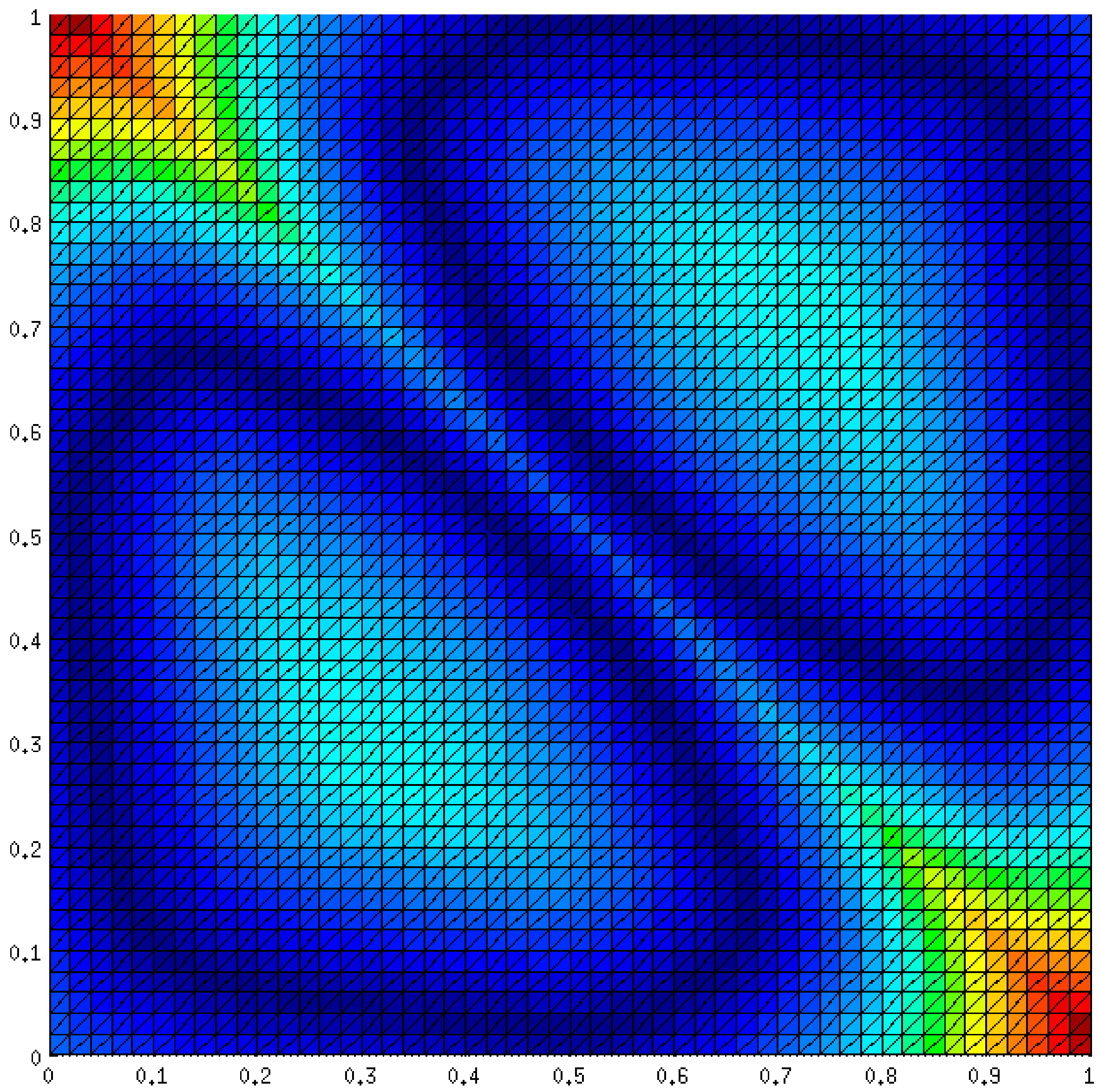

$|f(.) - g_\delta(.)|$ for $\delta = 1/50$

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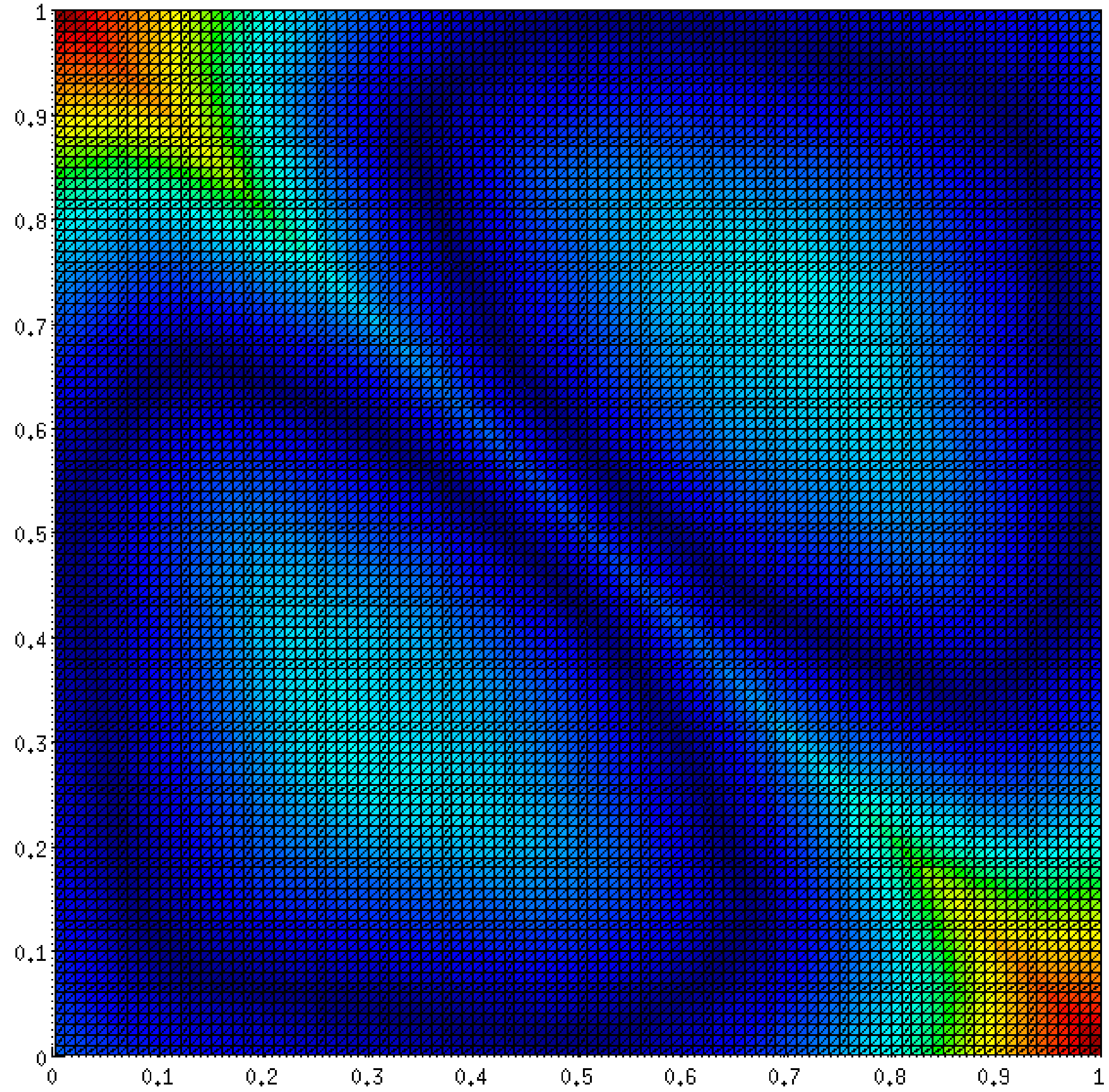

$|f(.) - g_\delta(.)|$ for $\delta = 1/100$

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Related recent work

Parametrization by "supporting planes" [Ekeland—Moreno-Bromberg '10]

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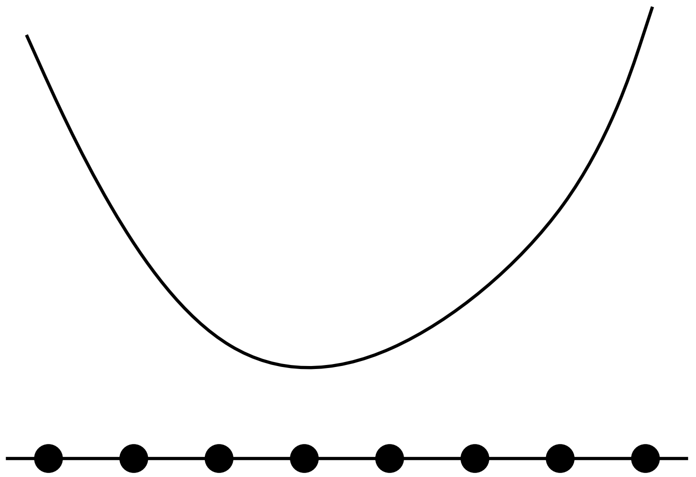

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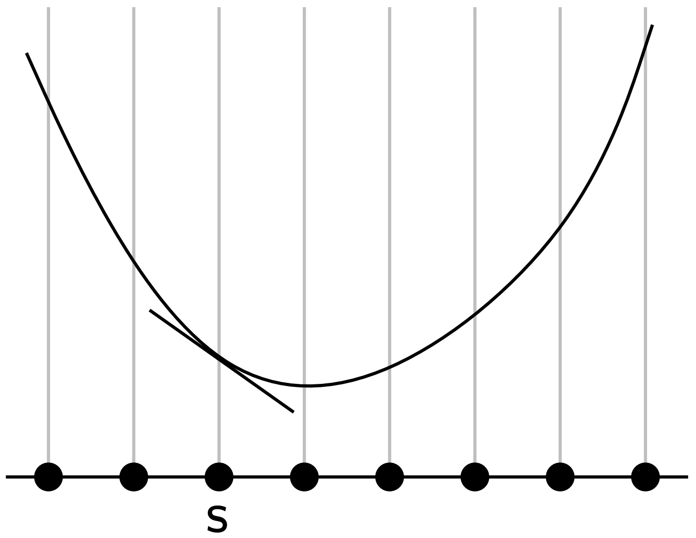

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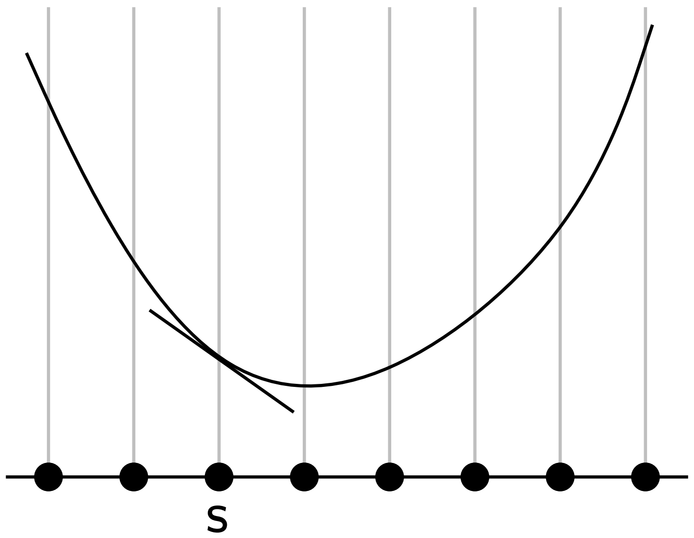

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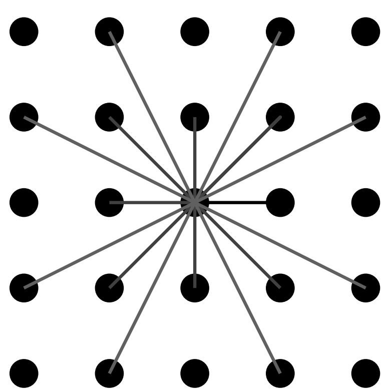

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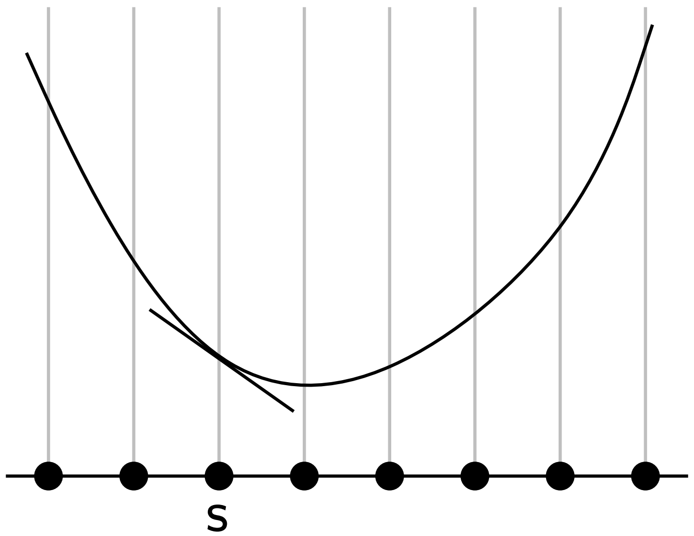

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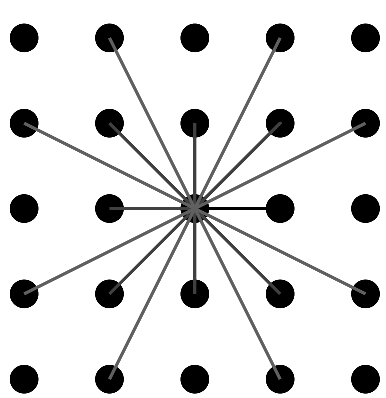

restricted to grids

no complete convergence theory

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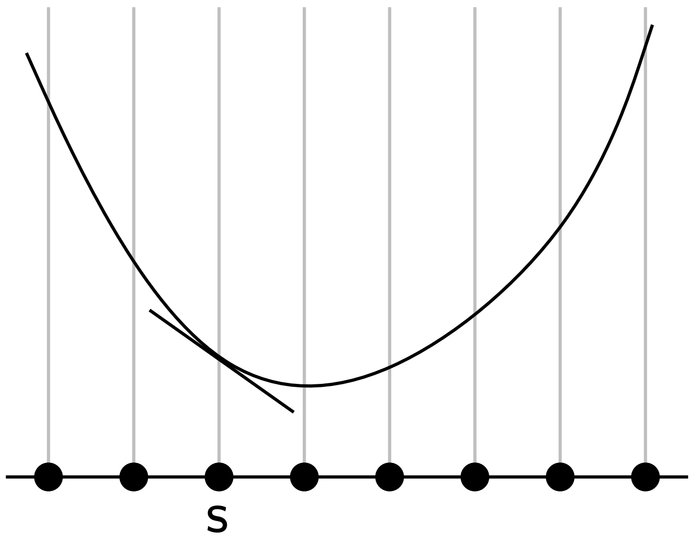

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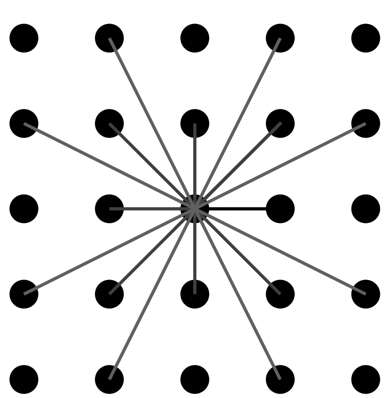

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Our goal: "external" discretization of convexity constraints for various space of functions (e.g. finite-element of order 1,2,3, tensor-product splines, etc.)

1. Discretization of convexity constraints

Relaxation of (convexity) constraints

Definition: given a metric space X and $L \subseteq \mathrm{Aff}(\mathcal{C}(X))$, define

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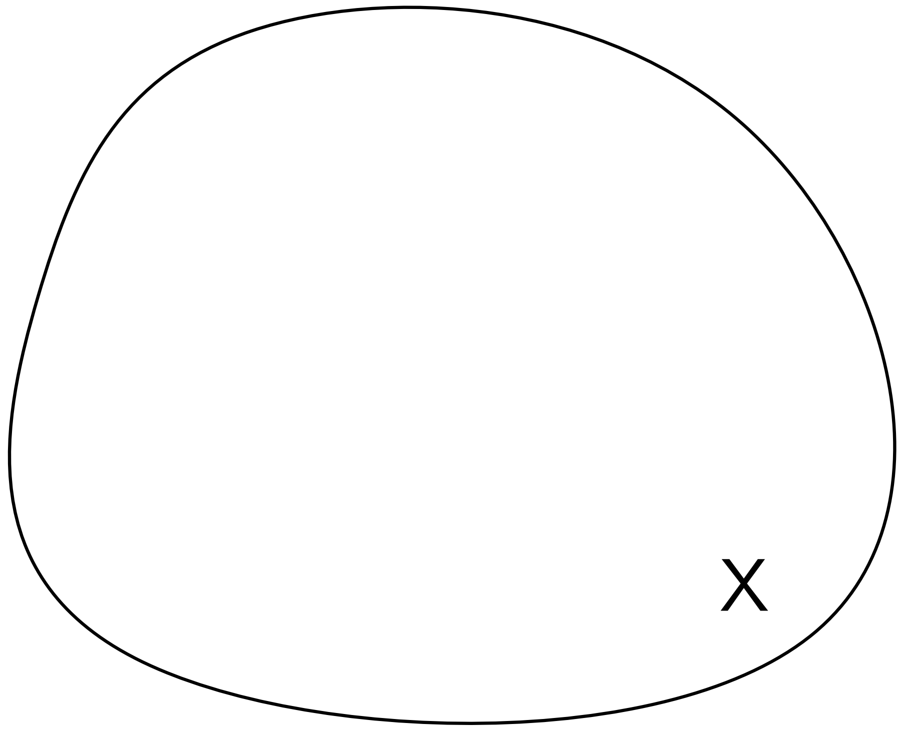
Proposition: Let M be an α-relaxation of L_2. Then,

$$\forall g \in \mathcal{H}_M,\ \exists \bar{g} \in \mathcal{H} \text{ s.t. } \|g - \bar{g}\| \leq d\alpha(g)$$

Discretization of convexity constraints



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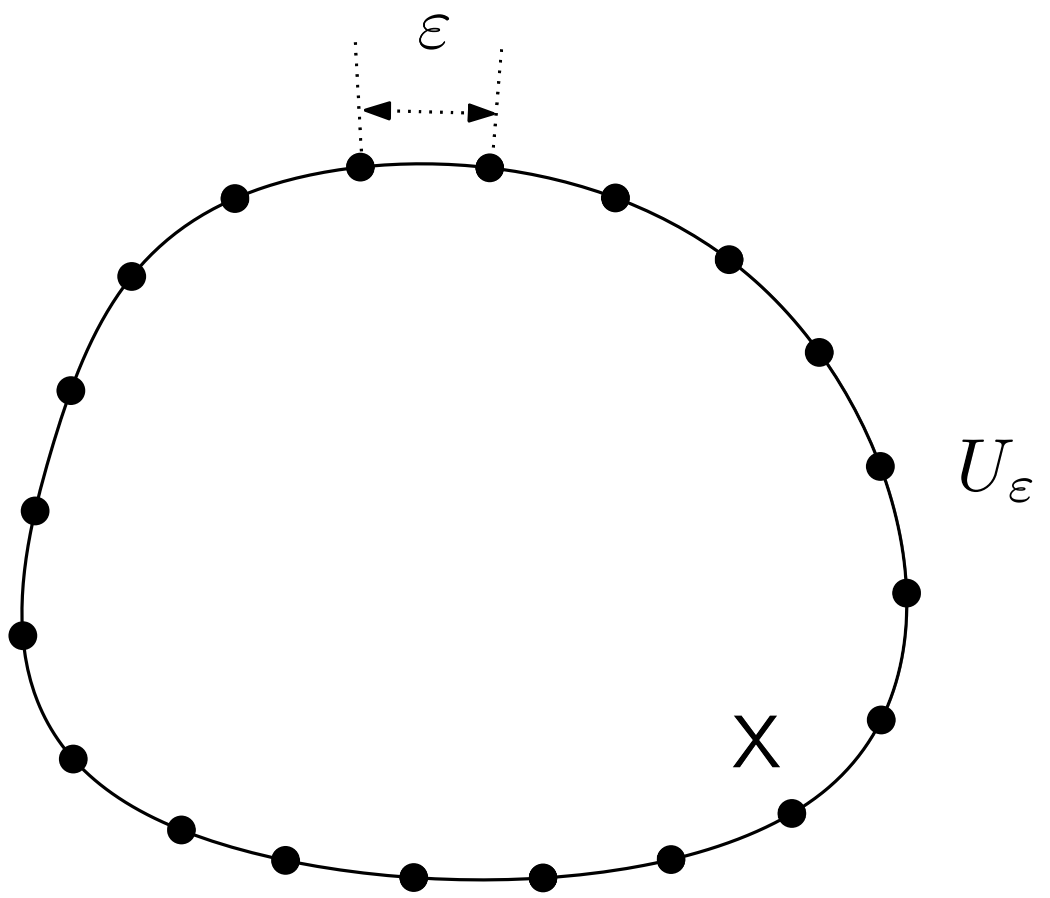

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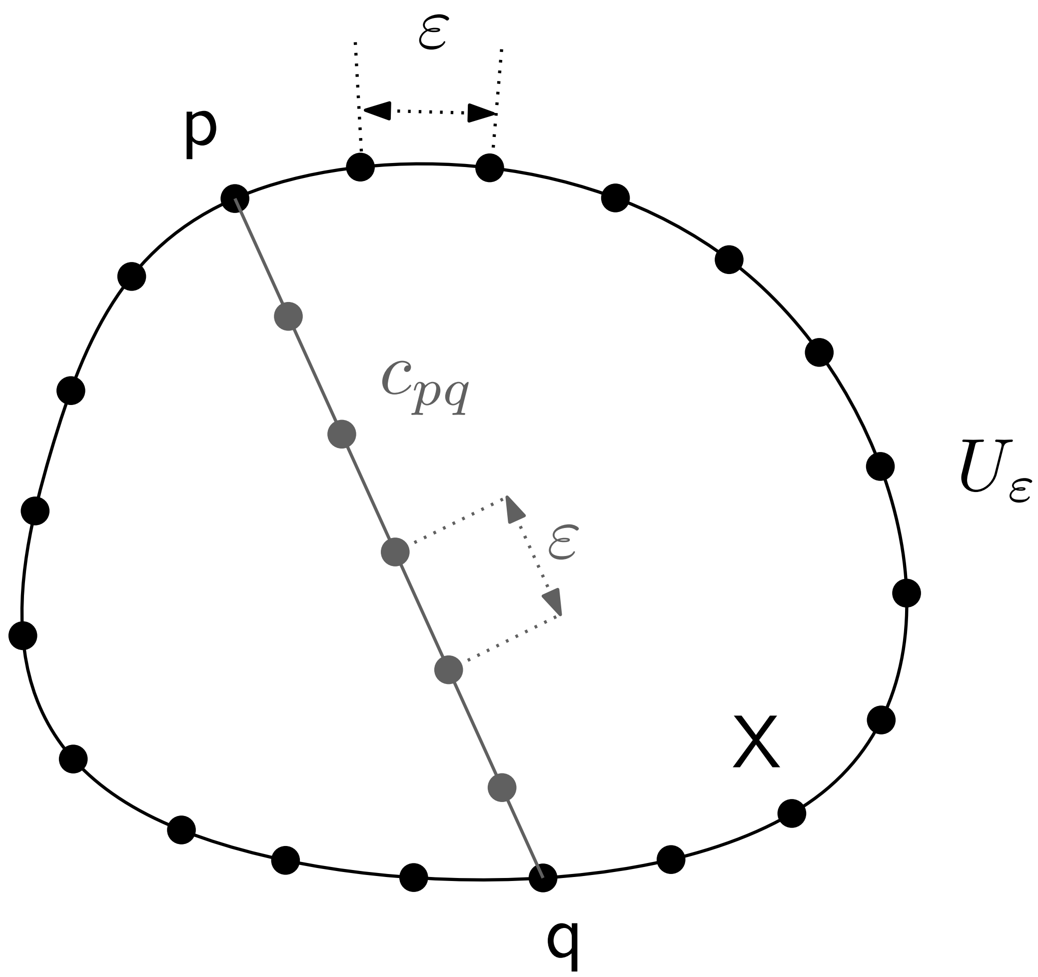

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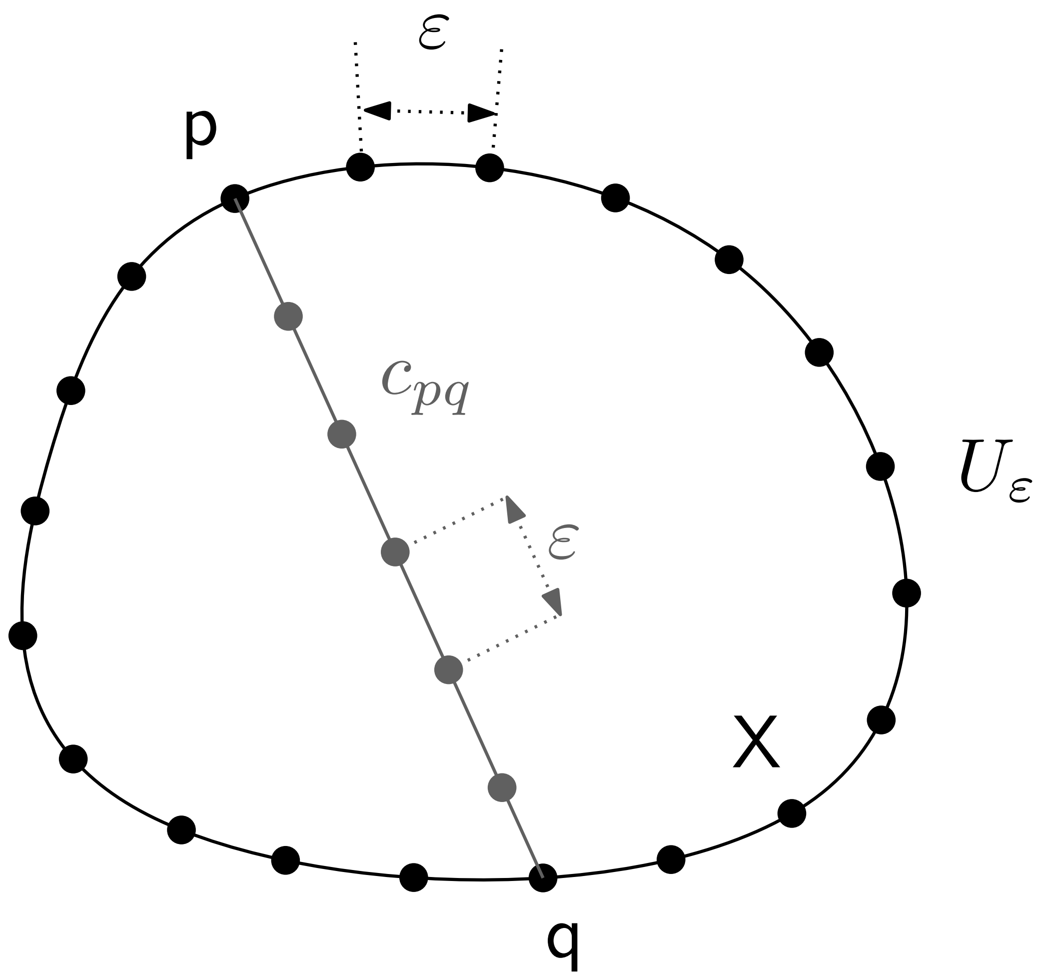

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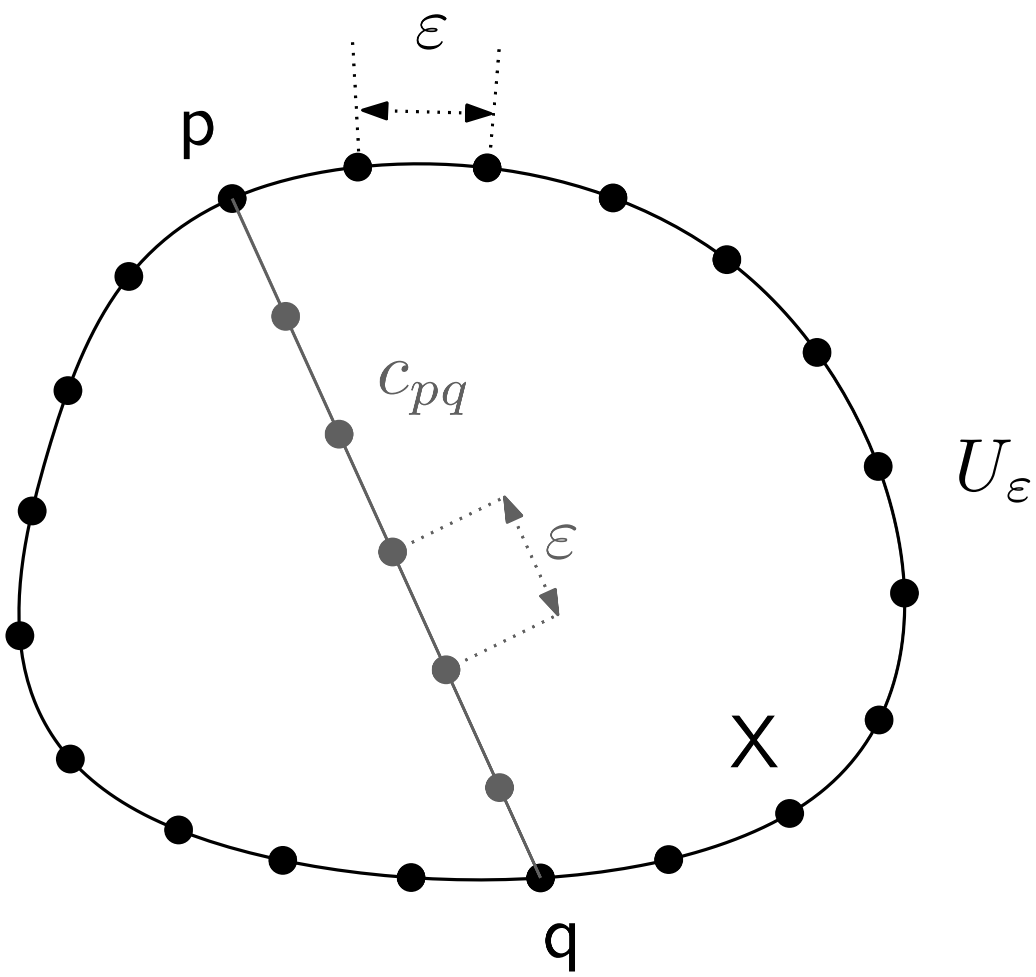
for $z \in [x, y]$,

$$\ell_{xyz}(g) = g(z) - \frac{\|yz\|}{\|xy\|} g(x) - \frac{\|xz\|}{\|xy\|} g(y)$$

Discretization of convexity constraints



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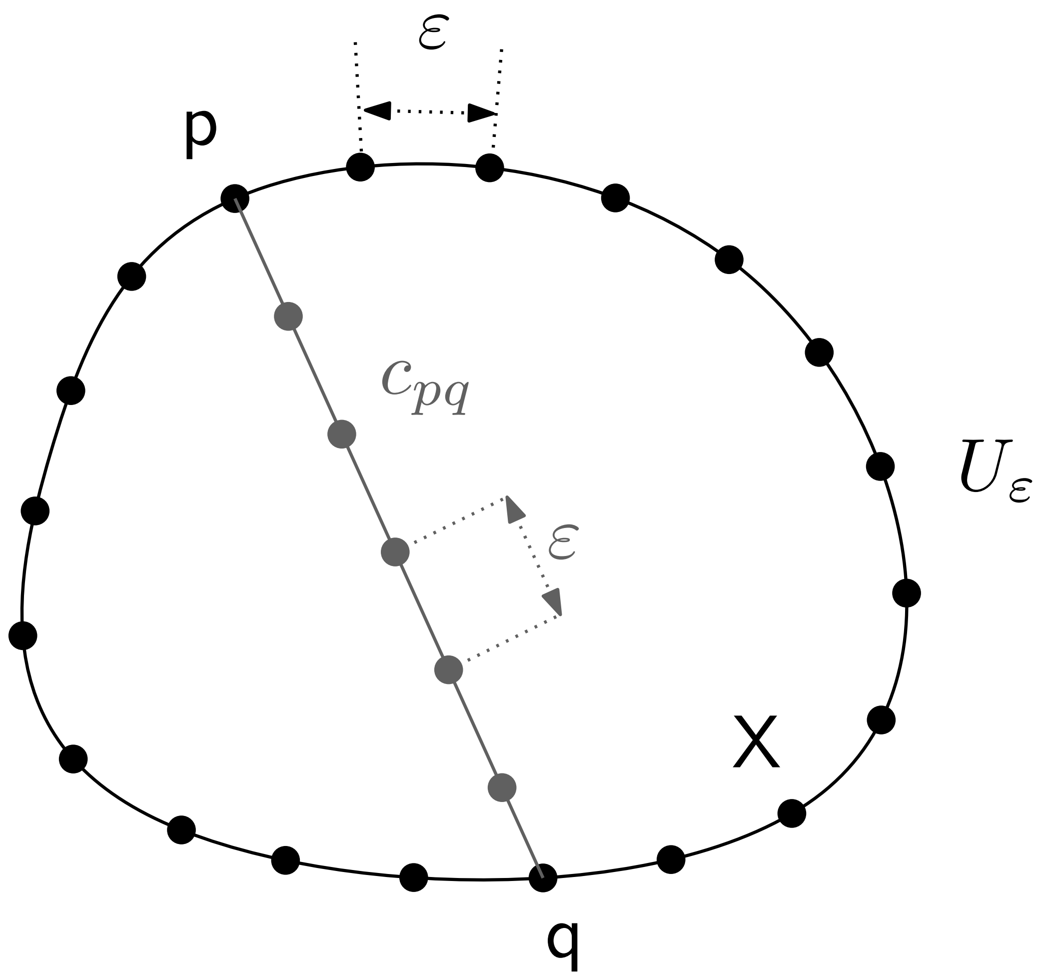

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Theorem: $\forall g \in \mathcal{H}_{M_\varepsilon^c}, \exists \bar{g} \in \mathcal{H}$ s.t. $\|g - \bar{g}\| \leq c_d \mathrm{Lip}(g)\varepsilon$.

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When is the finite-dimensional polyhedron $E_\delta \cap \mathcal{H}_{M_\varepsilon^c} \cap \mathrm{B}^\gamma_{\mathrm{Lip}}$ a "good approximation" of the convex set $\mathcal{H} \cap \mathrm{B}^\gamma_{\mathrm{Lip}}$?

Hausdorff approximation results

Definition: The (half) Hausdorff distance between $A, B \subseteq \mathcal{C}(X)$ is given by

$$\mathrm{h}_{\mathrm{H}}^{p}(A,B) = \min\left\{r \geq 0; \forall f \in A,\ \exists g \in B,\ \|f-g\|_{\mathrm{L}^p(X)} \leq r\right\}.$$

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Moreover, $\|f_\delta + \eta s_\delta - f\|_\infty = O(\gamma\delta + \gamma\delta/\varepsilon^2) = O(\gamma\delta^{1/3})$ with $\varepsilon = \delta^{1/3}$

2. Numerical details

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linear maps (matrices)

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$\min_x g_1(L_1 x) + \ldots g_m(L_m x)$, $x \in \mathbb{R}^N$

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For $n = 0, 1, \ldots$

$$x_n = Q^{-1} \sum_{i=1}^m L_i^T (y_{i,n} - z_{i,n})$$

For $i = 1, \ldots, m$ $\quad (\gamma > 0)$

$$s_{i,n} = L_i x_n$$
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$y_{i,n+1} = \operatorname{prox}_\gamma g_i(s_{i,n} + z_{i,n})$ — parallelizable

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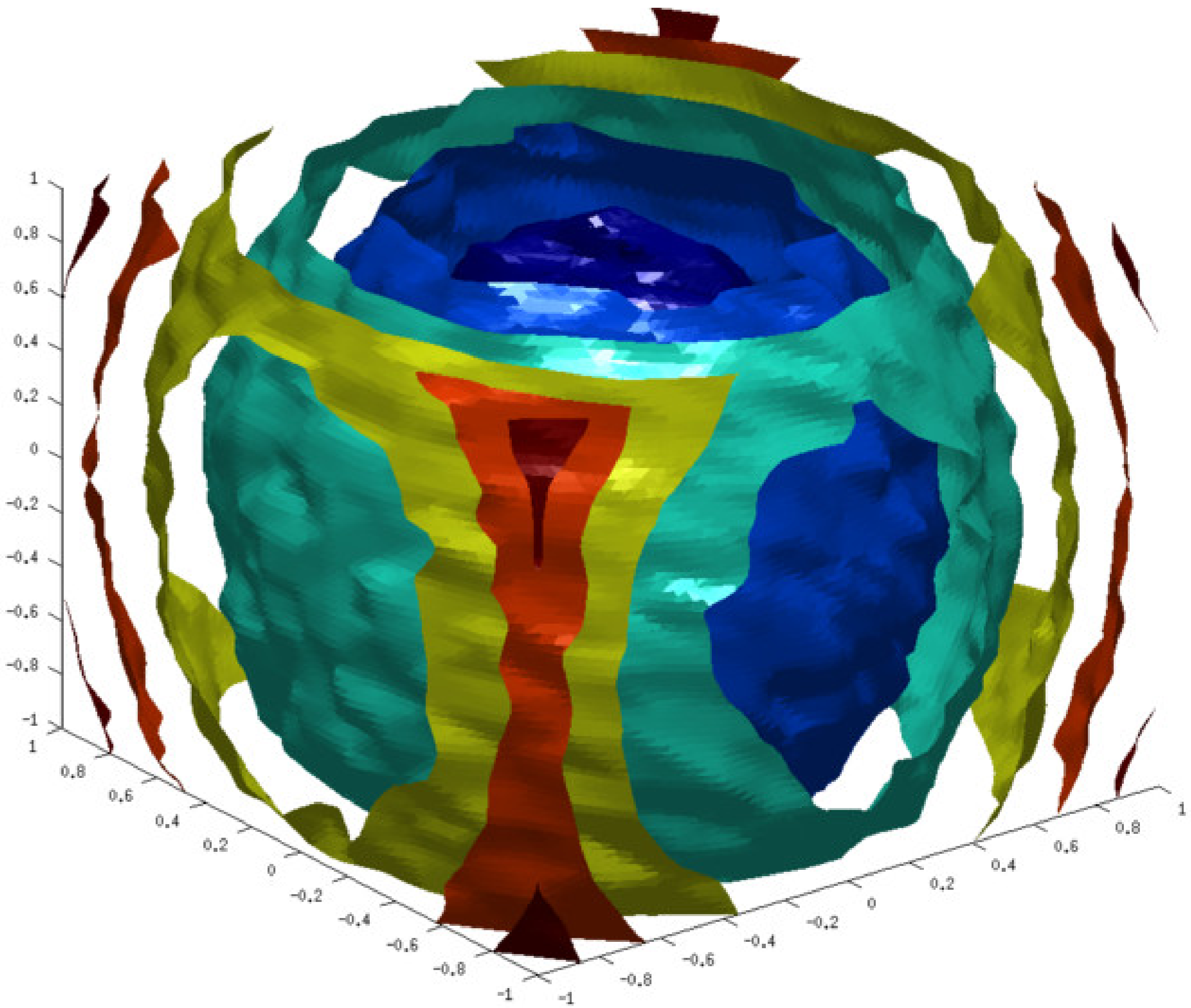
Meyer '99: simple and exact active set algorithm to solve this type of problems

Application: projection on convex functions

$X = [-1, 1]^3$

$u_0 : (x, y, z) \in X \mapsto \frac{x^2}{3} + \frac{y^2}{4} + \frac{z^2}{8}$



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Parameters: $\delta = \frac{1}{80}$ (80^3 grid)
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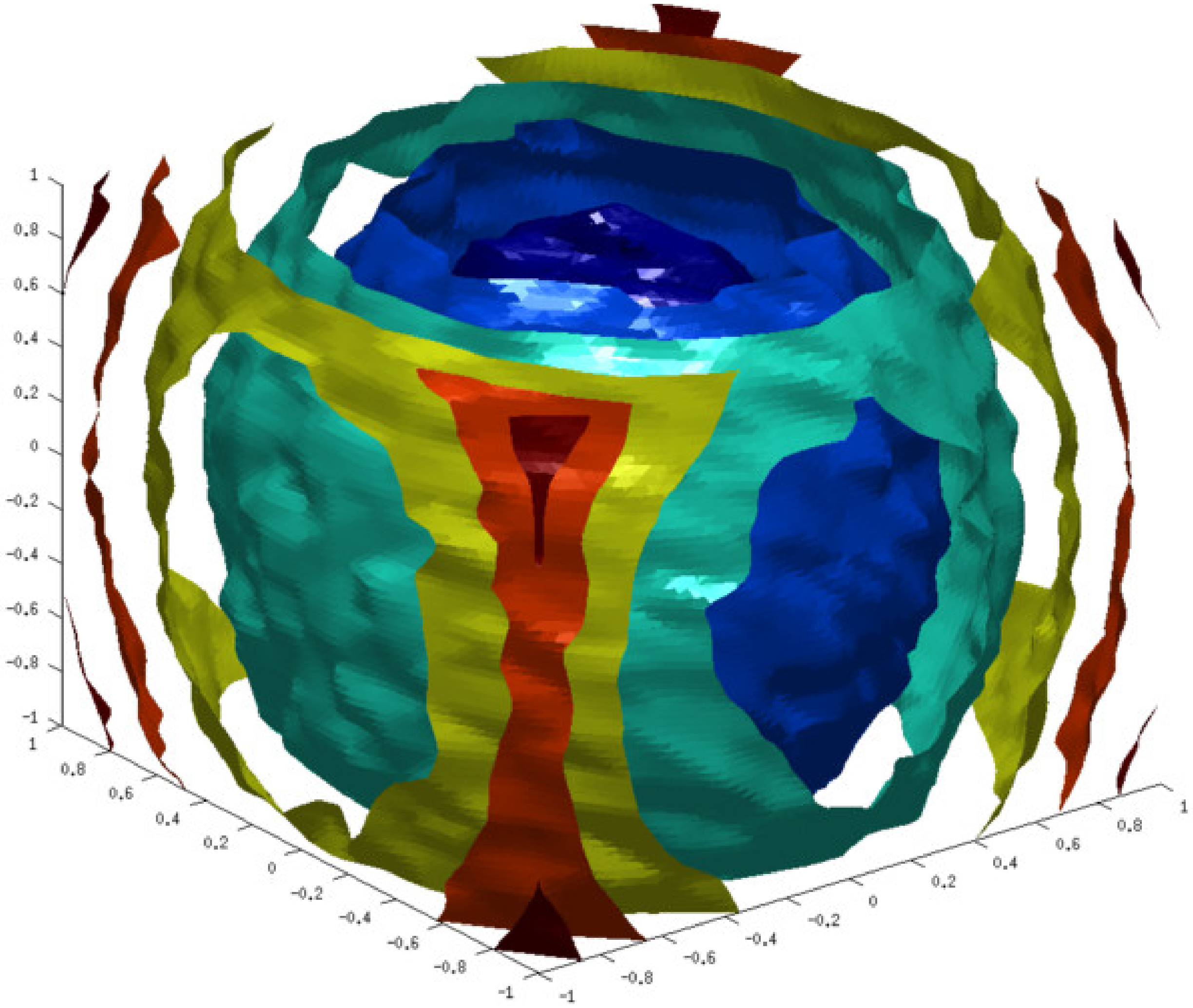

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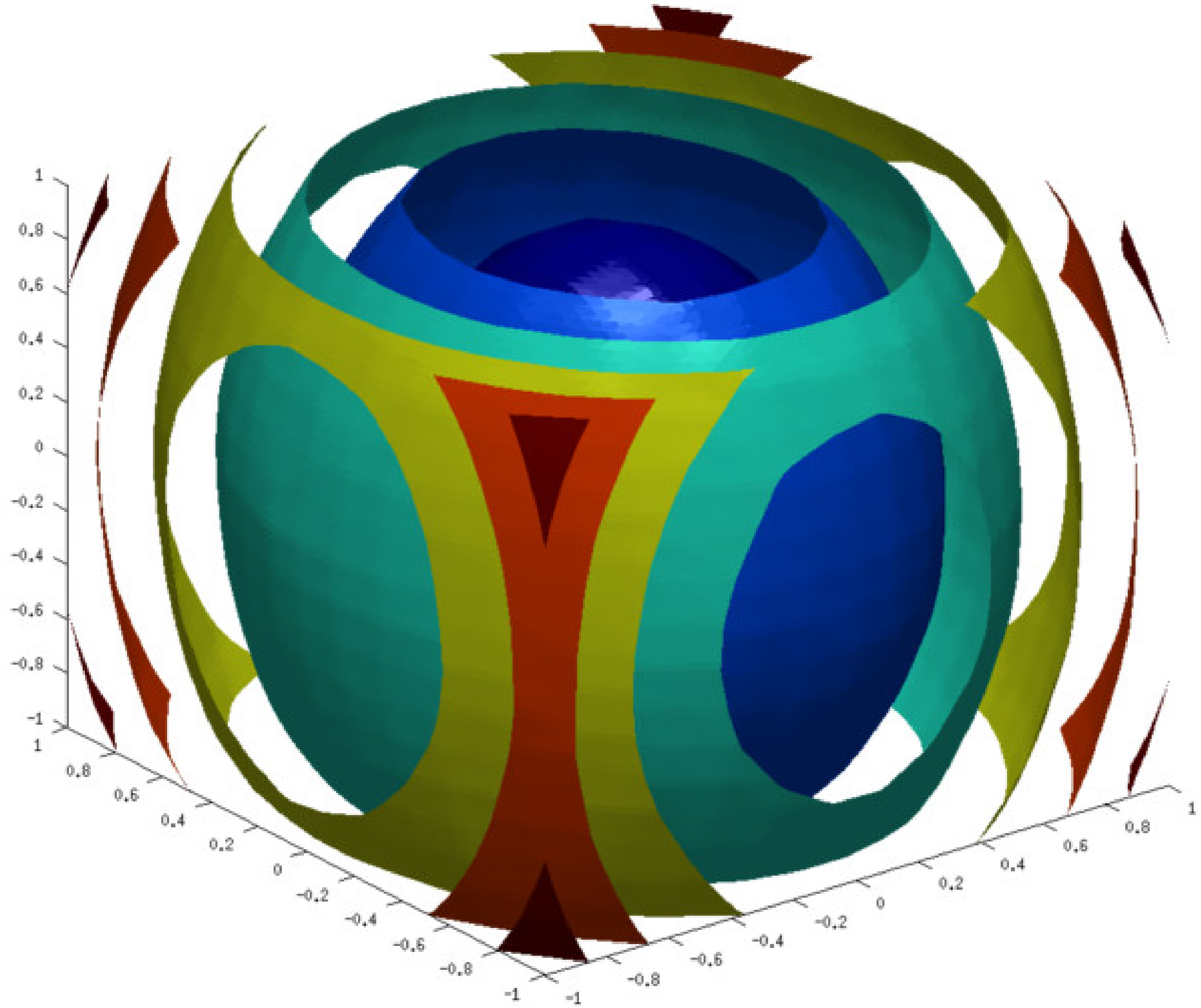

$$v \simeq \mathrm{proj}_{\mathcal{H}}(u)$$

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where $b = \|.\|^2$ is the cost of production

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$$\min_{u\in\mathcal{H},u\geq 0}\int_X \frac{1}{2}\|\nabla u(x)-x\|^2+u(x)\,\mathrm{d}\,x$$

Parameters: $\delta=\frac{1}{60}$ (60^2 grid)
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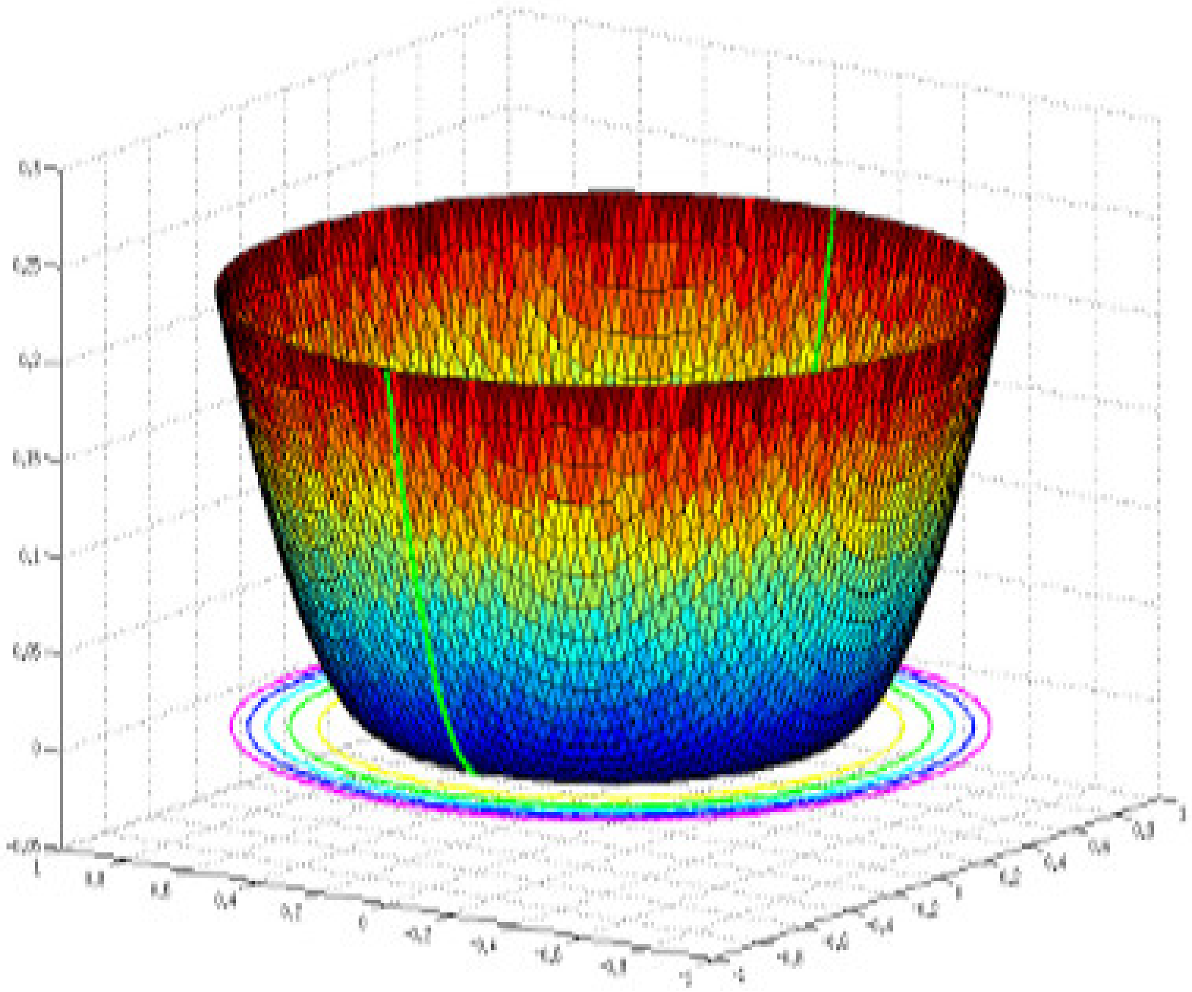

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in green, 1D solution (radial problem)

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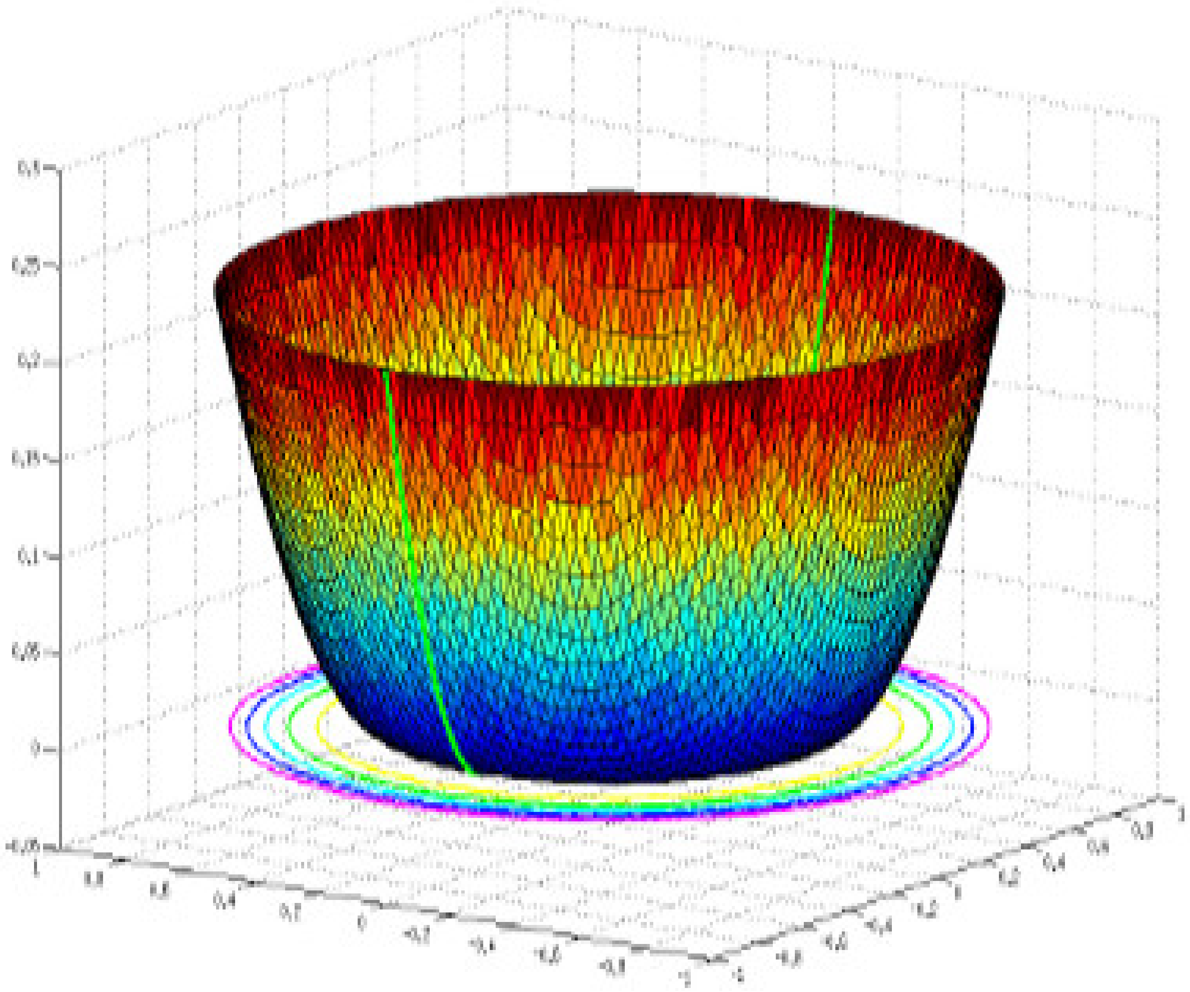

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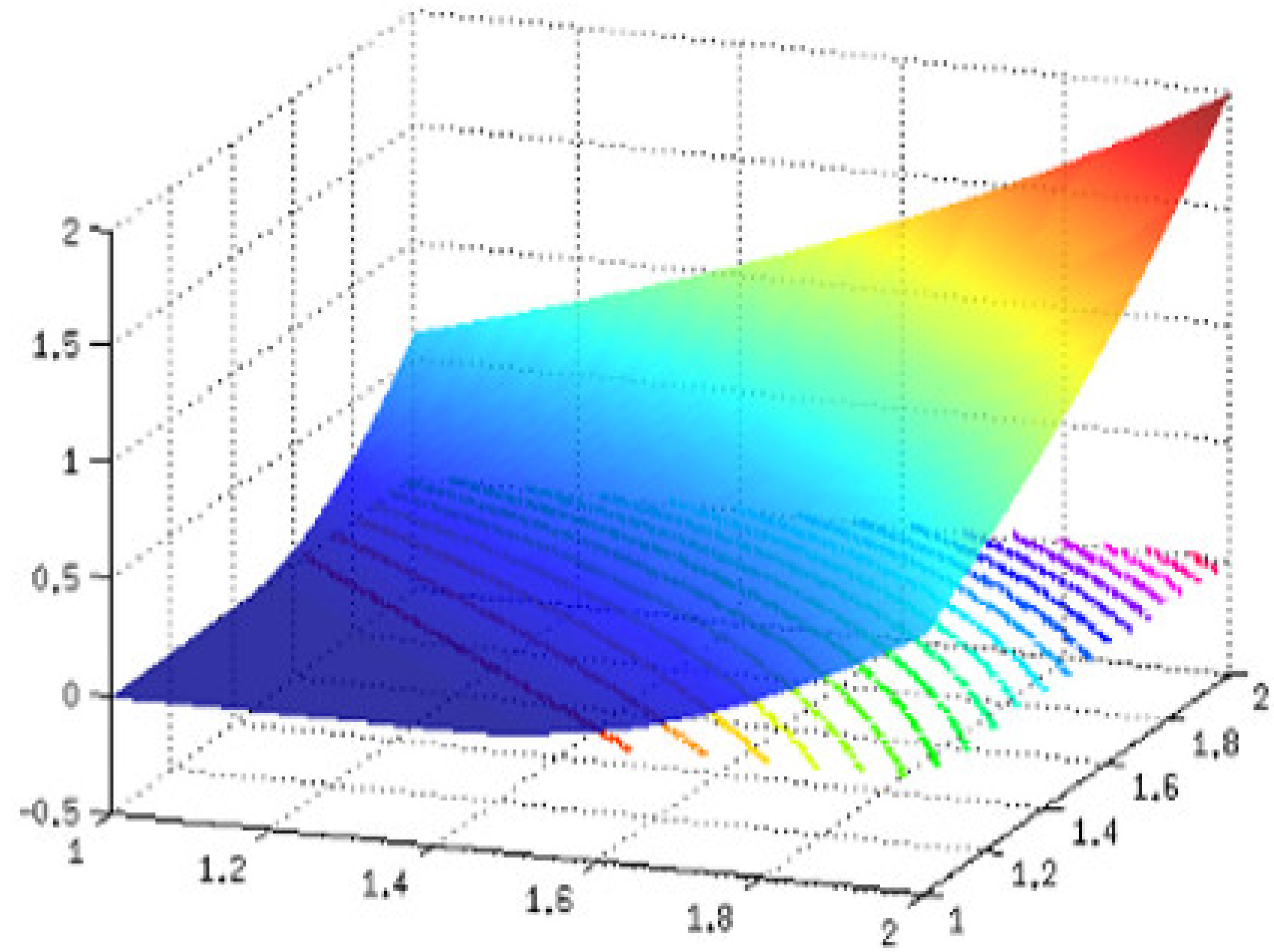
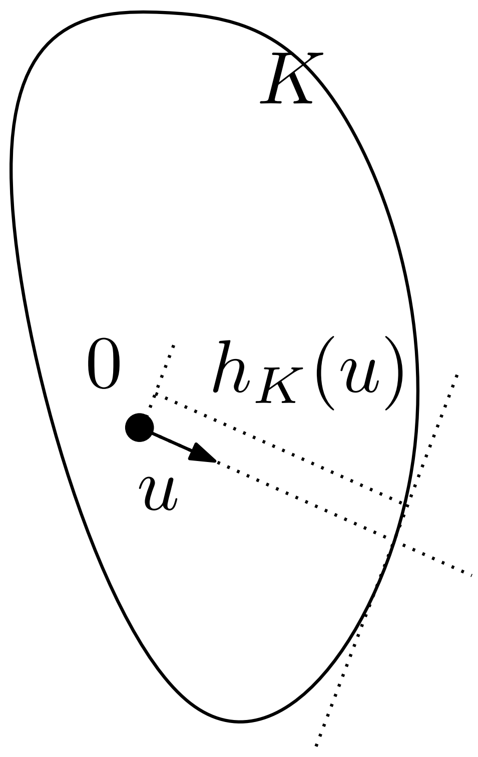


$\Omega=[1,2]^2$

3. Discretization of convexity-like constraints

Support functions of convex sets



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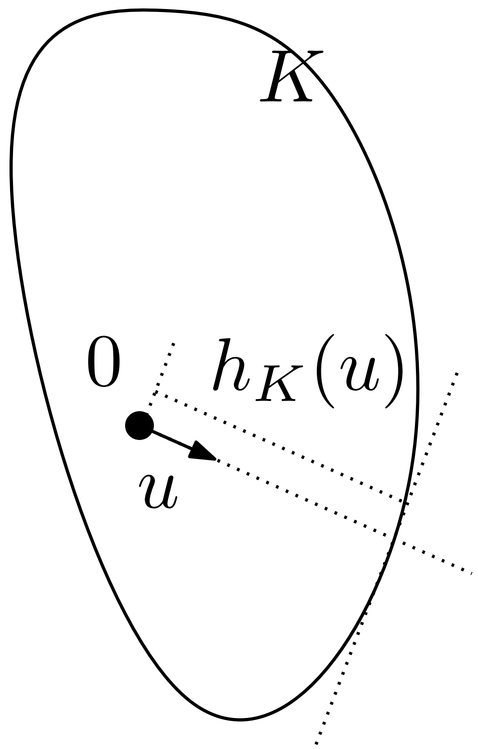

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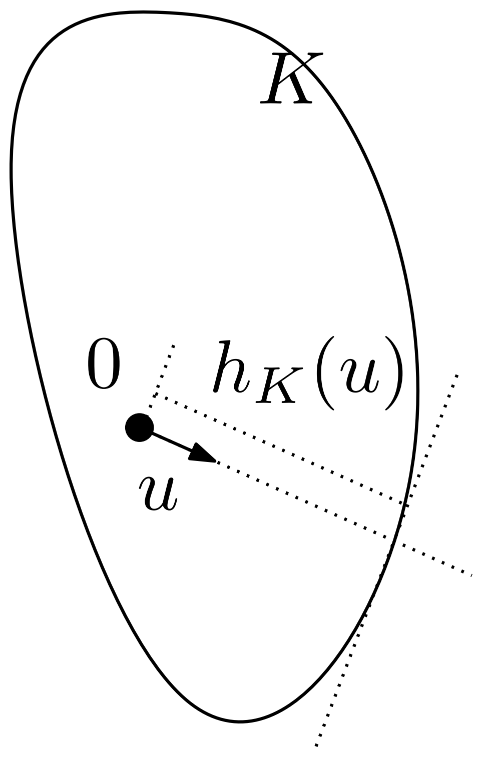

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$\iff \forall x, y \in \mathcal{S}^{d-1}$, $\forall\lambda \in [0,1]$, $z := \lambda x + (1-\lambda)y$,

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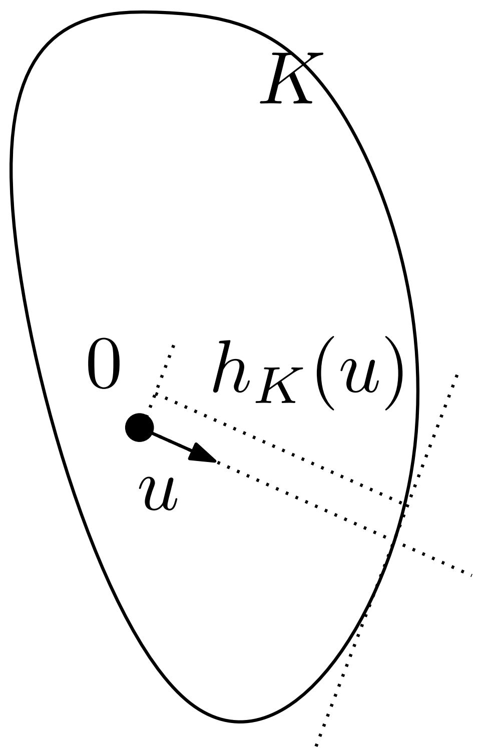

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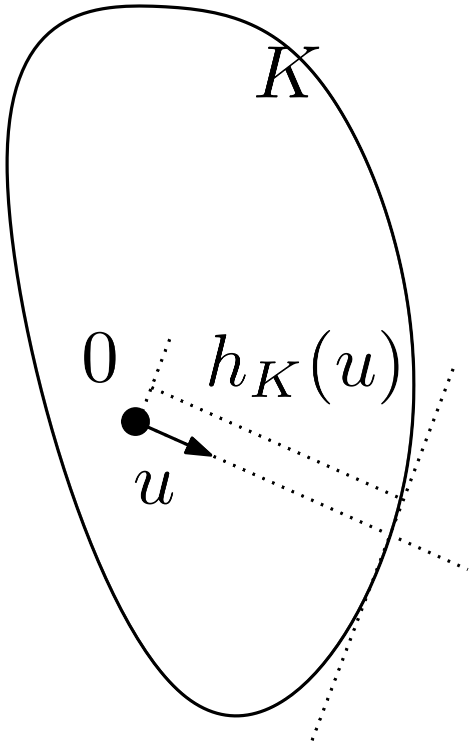

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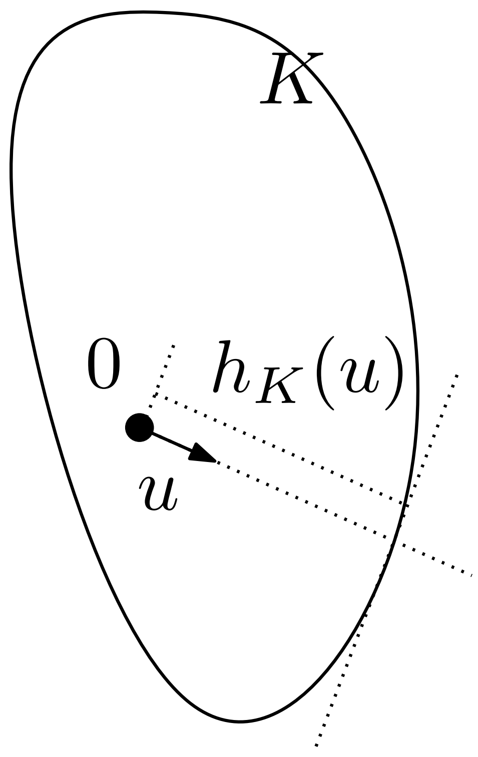

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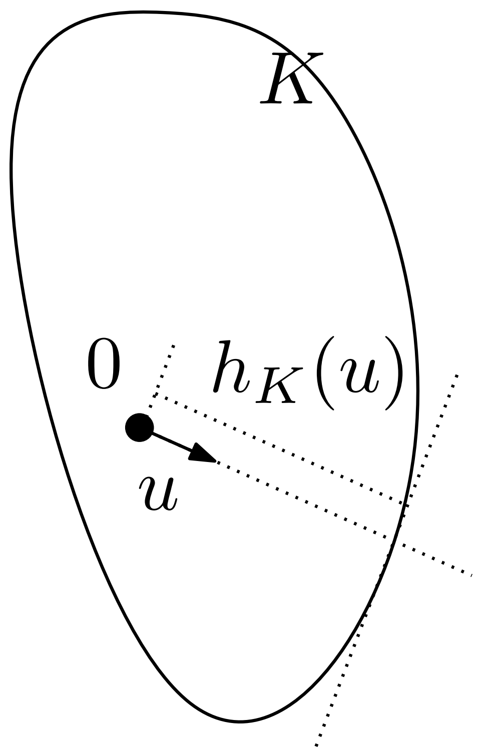

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... similar type of theorems as for the discretization of convexity constraints ...

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(ii) h is the support function of a convex set

$\iff \log(h)$ is c-convex for $c(x,y) = -\log(\langle x|y\rangle^+)$

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Theorem: Assuming (A0)–(A2), $\mathcal{H}_c$ is convex iff (NNCC) holds.

[Figalli, Kim, McCann '11]

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(A2) the sets X_{y_0} and Y_{x_0} are convex.

(NNCC) $\forall (y_0, y) \in Y$, $v \in X_{y_0} \mapsto c(\exp^c_{y_0} v, y_0) - c(\exp^c_{y_0} v, y)$ is convex, where $\exp^c_{y_0}$ is the inverse of the map $x \mapsto -\nabla_y c(x, y_0)$.

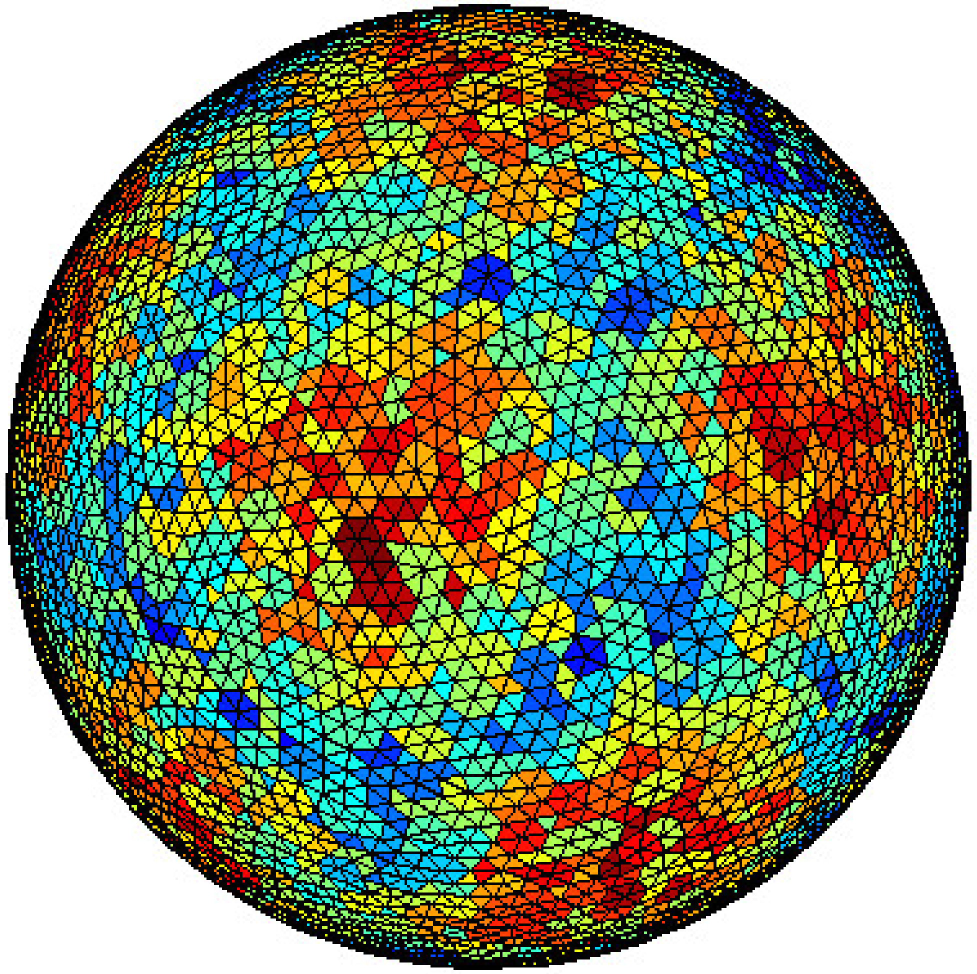
Proposition: Assuming (A0)–(A2) and (NNCC), ϕ belongs to $\mathcal{H}_c$ iff

$$\forall y \in Y, \ \phi_y : v \in X_y \mapsto \phi(\exp^c_y v) + c(\exp^c_y v, y) \in \mathcal{H}$$

Application: projection on support functions

$X = [-1, 1]^3$

$h_0 =$ support function of unit icosaedron



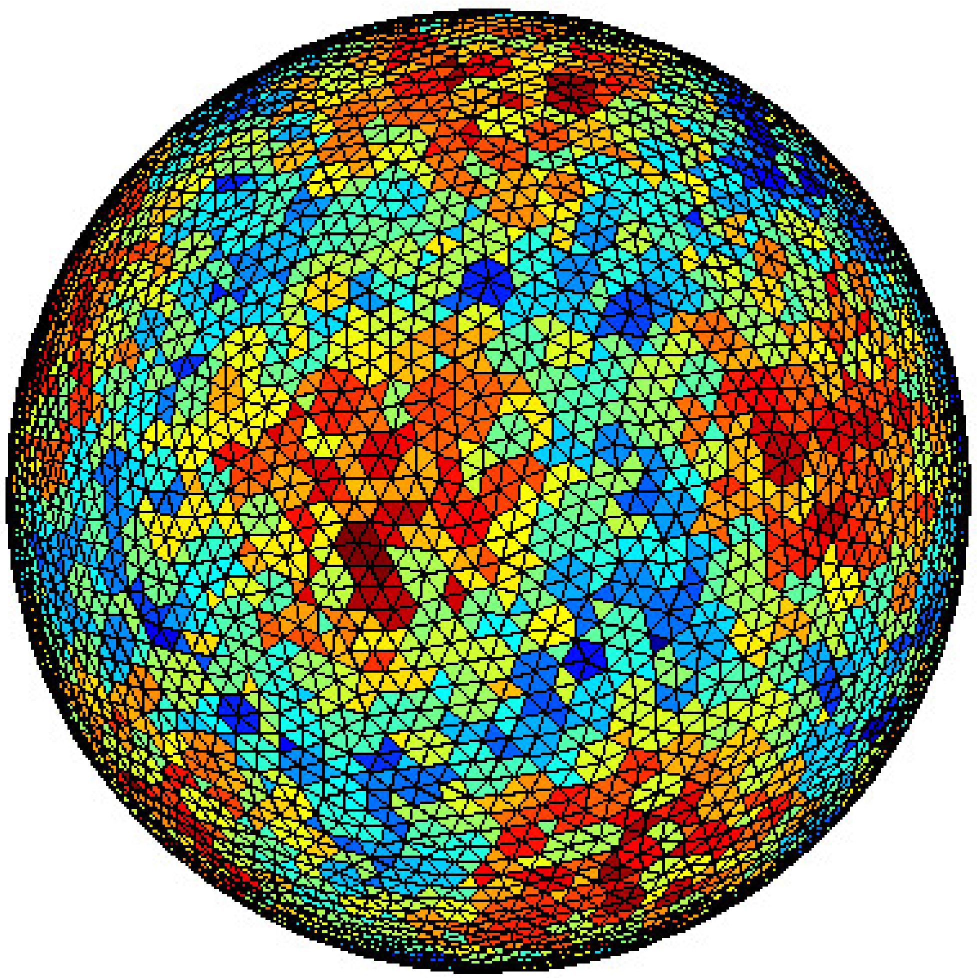
$h = h_0 + \frac{1}{40}\mathcal{N}(0, 1)$

Application: projection on support functions

$X = [-1, 1]^3$

$h_0 =$ support function of unit icosaedron

Parameters: $\delta \simeq 1/20$ ($5k$ pts)
$\varepsilon = \frac{1}{50}$
$N_{\text{iter}} = 10^4$



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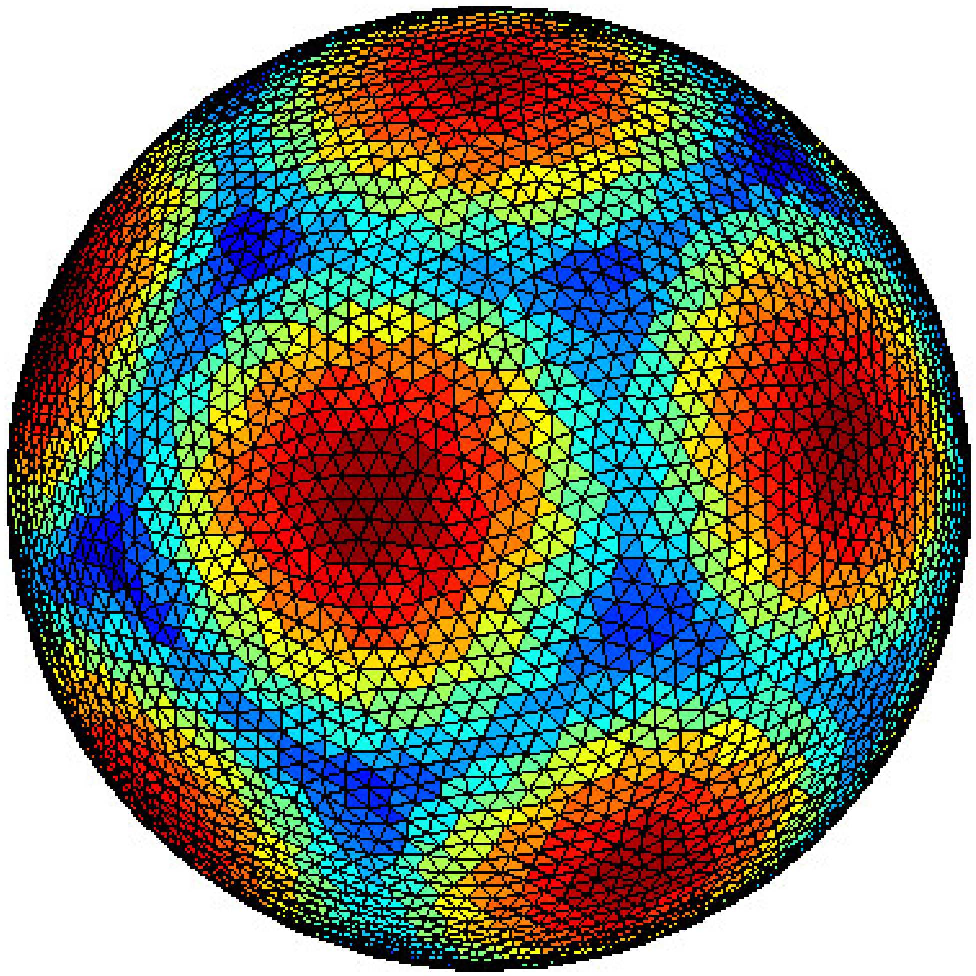


$\simeq \text{proj}_{\mathcal{H}^s}(h)$

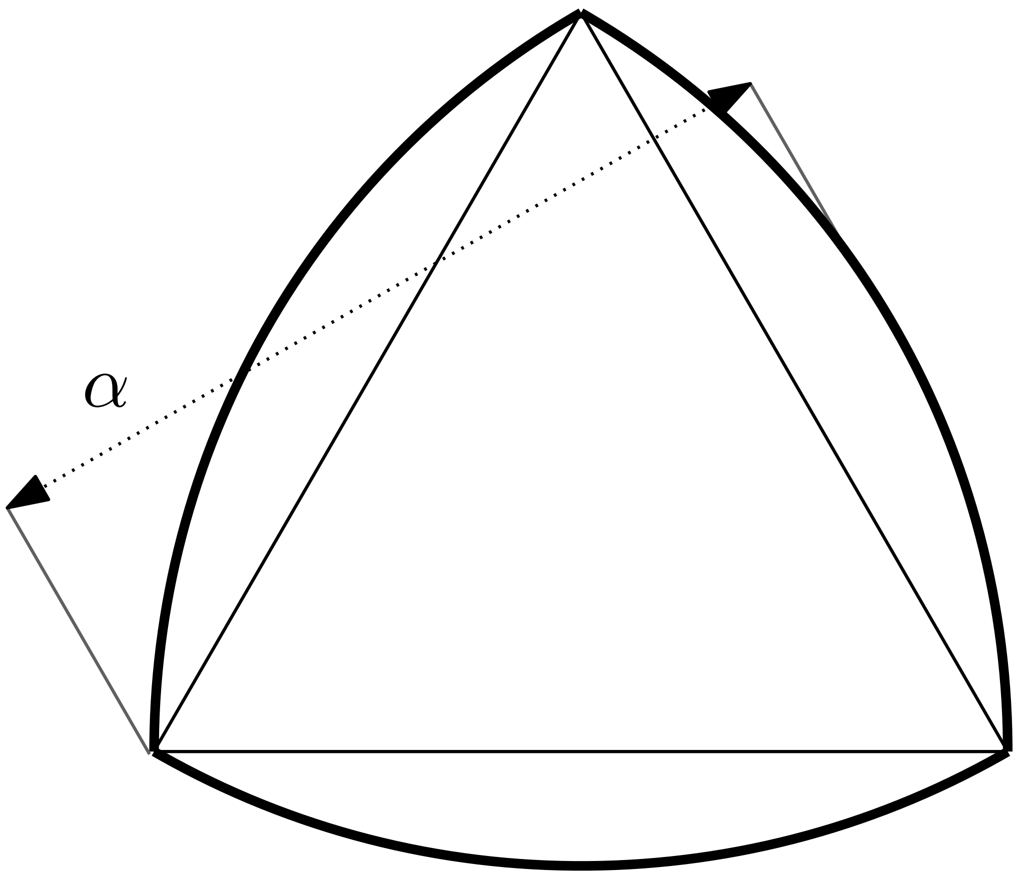
Application: convex bodies with constant-width

Definition: The **width** of K in direction u is

$$w_K(u) = h_K(u) + h_K(-u).$$

K has constant width α if $w_K(u) = \alpha$.

Application: convex bodies with constant-width



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Theorem: Reuleaux triangles minimize the volume over convex sets of the plane with constant width α.

[Blaschke-Lebesgue]

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Theorem: Reuleaux triangles minimize the volume over convex sets of the plane with constant width α.

[Blaschke-Lebesgue]

Bonnensen-Fenchel conjecture (3D):

Meissner's body minimize the volume among convex sets with fixed constant width.

Application: convex bodies with constant-width

$h =$ support function of unit tetrahedron

Parameters: $\delta \simeq 1/20$ ($5k$ pts)
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Application: convex bodies with constant-width

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Parameters: $\delta \simeq 1/20$ ($5k$ pts)
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	Surface	Volume	Width	Relative width error
L^1 projection of h	2.6616	0.36432	0.951	$\leq$ 0.001
L^2 projection of h	2.5191	0.34312	0.920	$\leq$ 0.003
L^∞ projection of h	2.1351	0.28081	0.835	$\leq$ 0.001

Application: convex bodies with constant-width

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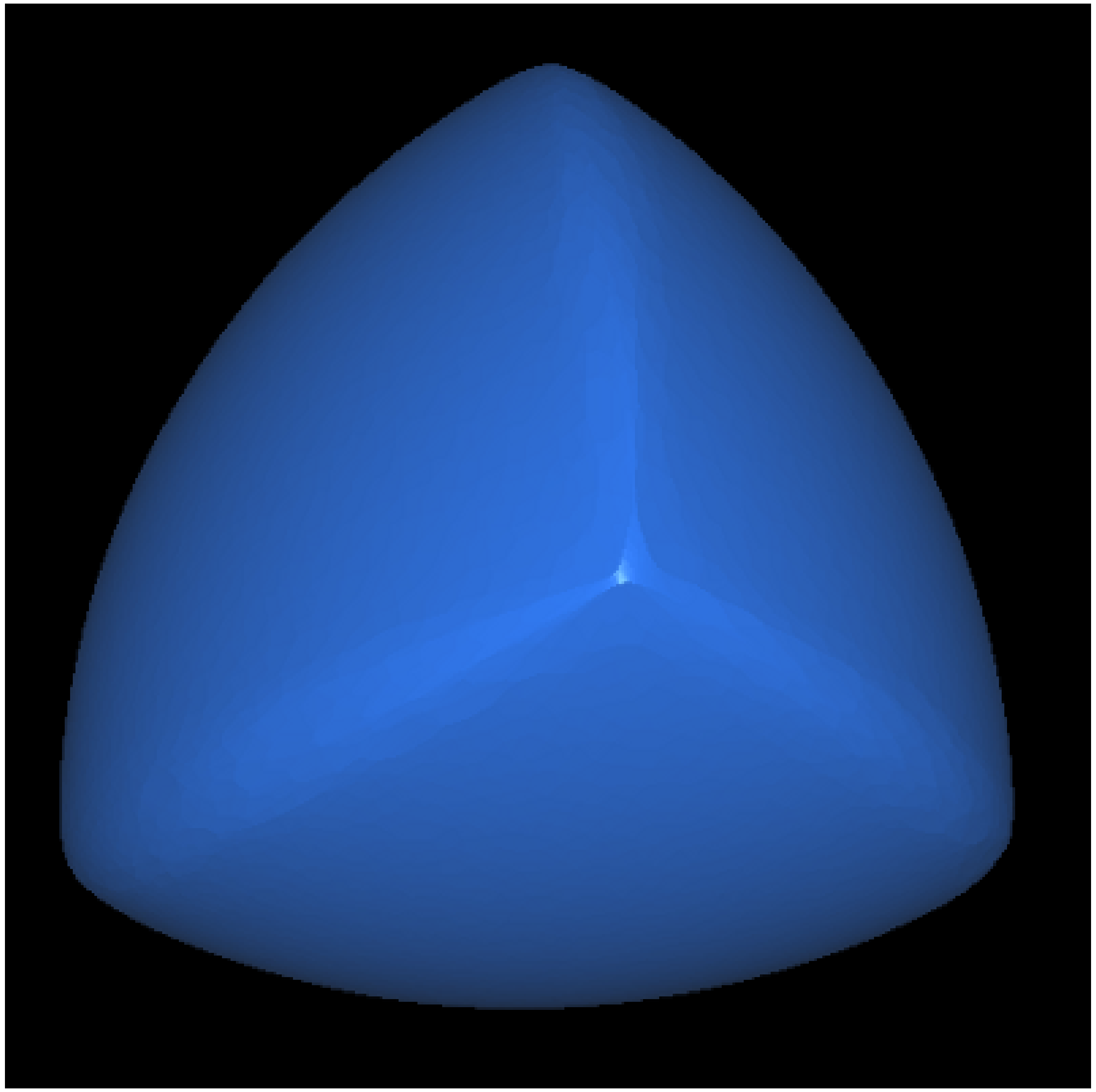

$\simeq \mathrm{L}^2$ projection of h on constant width

Application: convex bodies with constant-width

$h =$ support function of unit tetrahedron

Parameters: $\delta \simeq 1/20$ ($5k$ pts)
$\varepsilon = \frac{1}{50}$
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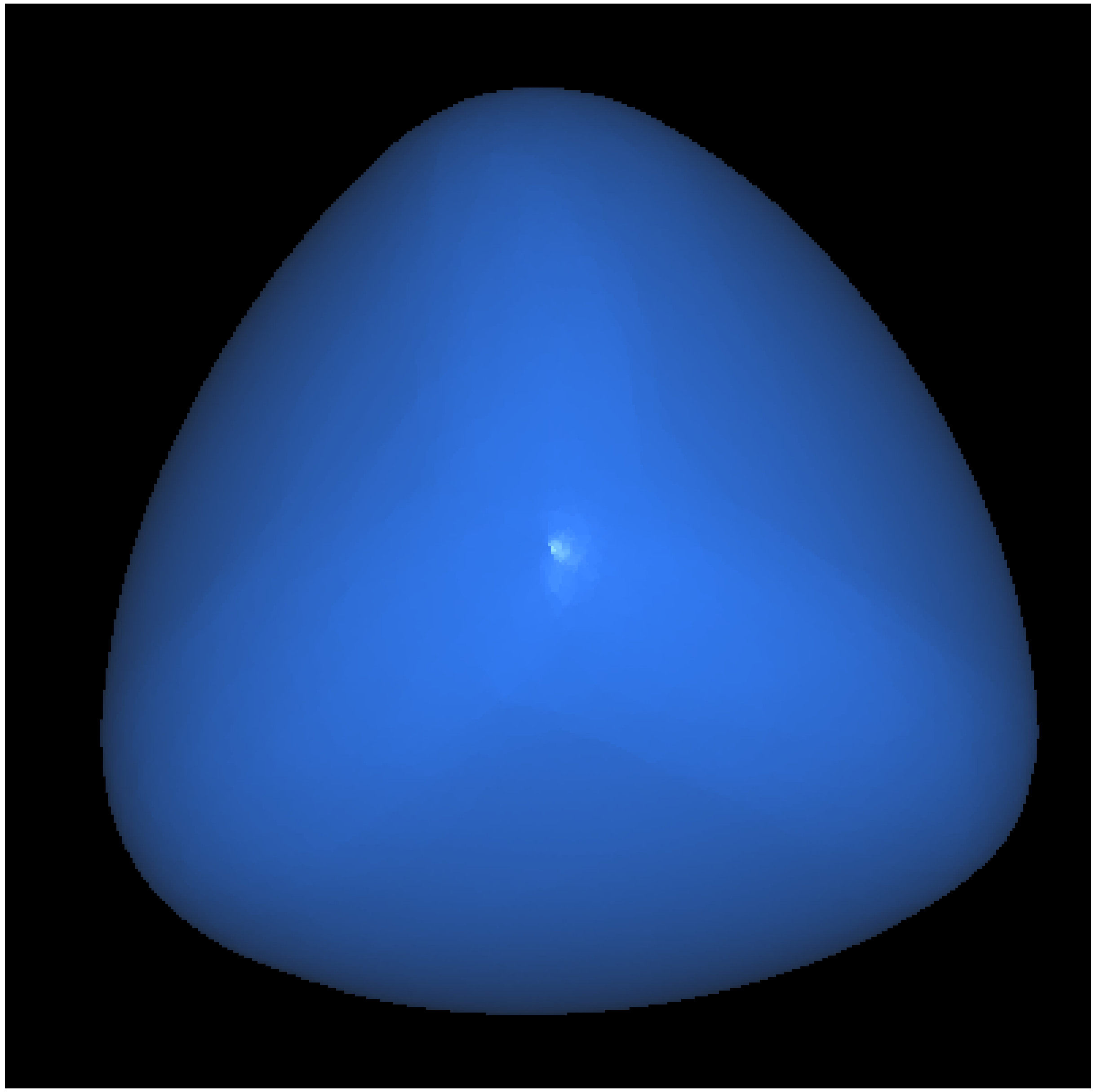
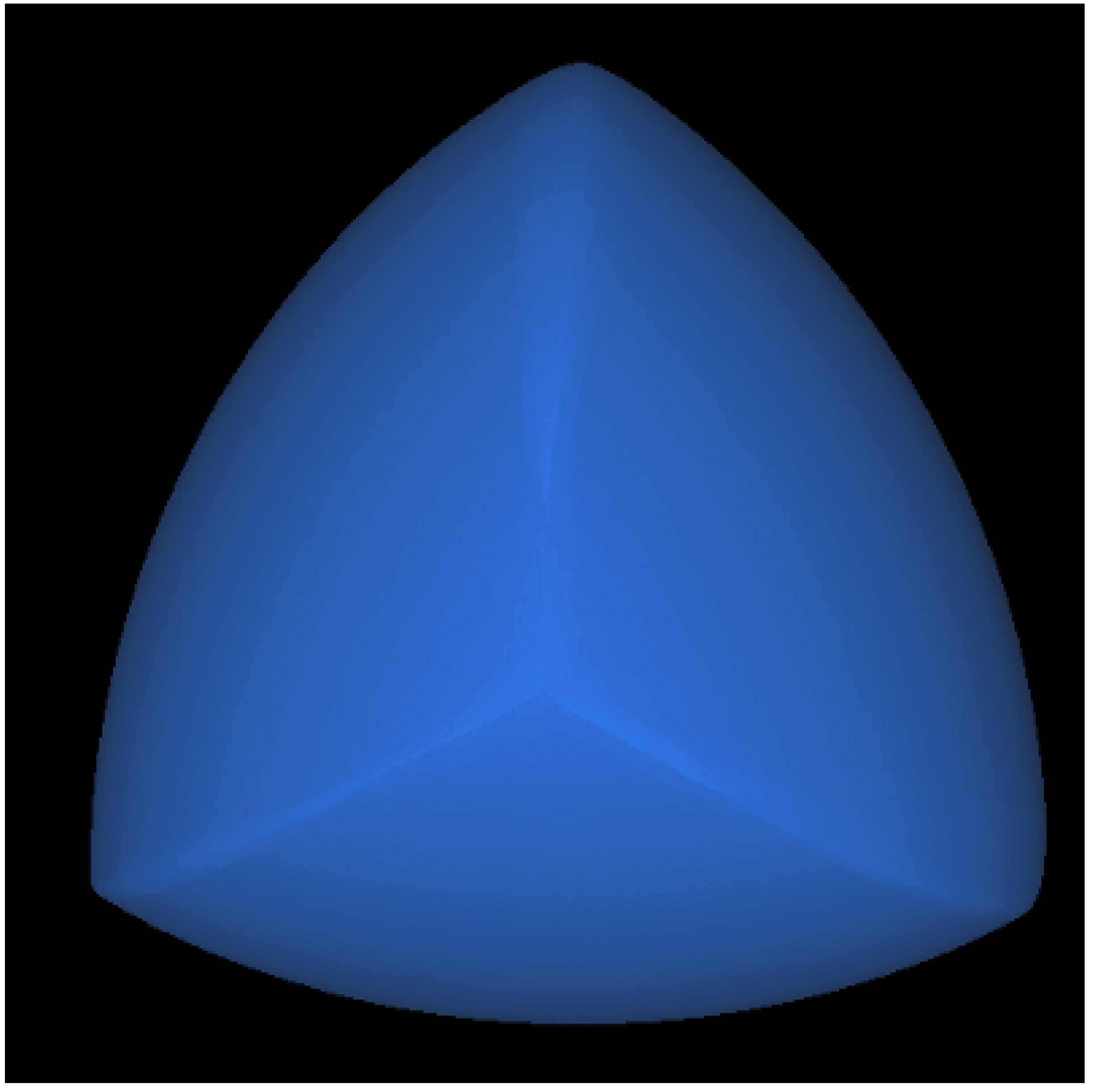


$\simeq \mathrm{L}^\infty$ projection of h on constant width

Application: convex bodies with constant-width

$h =$ support function of unit tetrahedron

Parameters: $\delta \simeq 1/20$ ($5k$ pts)
$\varepsilon = \frac{1}{50}$
$N_{\text{iter}} = 10^4$

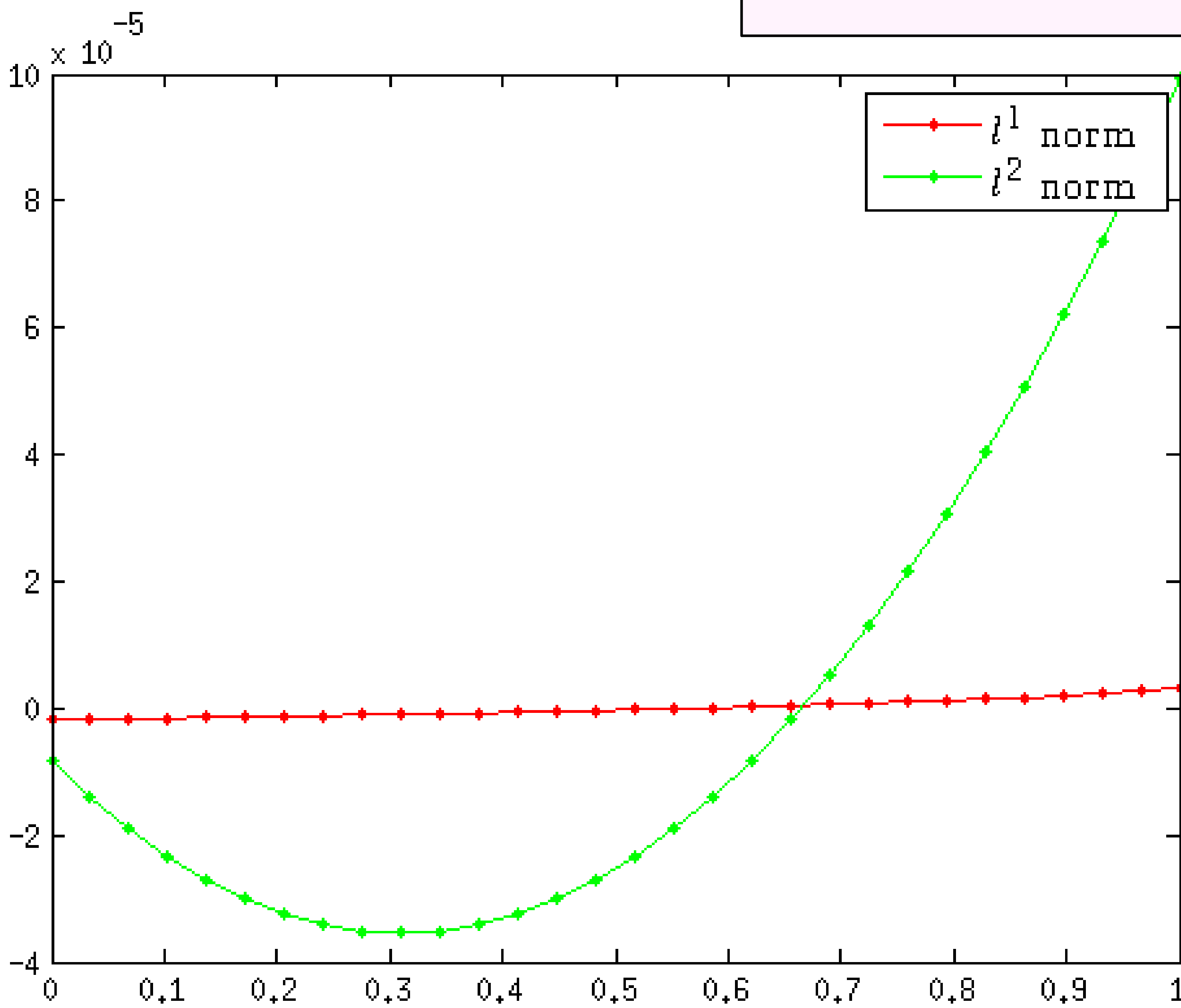


$\simeq \mathrm{L}^1$ projection of h on constant width

Application: convex bodies with constant-width

$h =$ support function of unit tetrahedron

Parameters: $\delta \simeq 1/20$ ($5k$ pts)
$\varepsilon = \frac{1}{50}$
$N_{\text{iter}} = 10^4$



distance between Δ^3 and the interpolation of the two Meissner bodies