Free surface ocean

$$D_t u + f e_3 \times u + \nabla p + \rho g e_3 = 0$$
$$D_t \rho = 0$$
$$\nabla . u = 0$$

on $[0, \tau) \times \Omega(t)$

$\Omega(t) = \Omega_2 \times [0, h(t, x_1, x_2)] \subset \mathbb{R}^3$; $\Omega_2 \subset \mathbb{R}^2$

$u \parallel \partial\Omega(t) \setminus \{x_3 = h\}$

$$\partial_t h + u_1 \frac{\partial h}{\partial x_1} + u_2 \frac{\partial h}{\partial x_2} = u_3$$

$p(t, x_1, x_2, h(t, x_1, x_2)) = p_h$ a constant

$u(0,x) = u_0(x)$, $\rho(0,x) = \rho_0(x)$, $h(0,x) = h_0(x)$

Scaled equations

$$\epsilon = \frac{U}{fL} = \left(\frac{H}{L}\right)^2 \quad \text{rotation dominated limit}$$

$$\epsilon D_t u_{\{12\}} + (-u_2, u_1) + \nabla_{\{12\}} p = 0$$

$$\epsilon^2 D_t u_3 + \nabla_3 p + \rho = 0$$

other equations unchanged

Semi-geostrophic equations

Define $\nabla_{\{12\}} p + (-u_{g2}, u_{g1}) = 0$, $u_{g3} = 0$

then $\nabla . u_g = 0$, $u = u_g + O(\epsilon)$

$$\epsilon D_t u_g + (-u_2, u_1) + \nabla_{\{12\}} p = O(\epsilon^2)$$
$$\nabla_3 p + \rho = O(\epsilon^2)$$

other equations unchanged

Energy conservation (Euler)

$$E = \int_{\Omega(t)} \left(\frac{1}{2} \left(\epsilon^2 u_{\{12\}}^2 + \epsilon^4 u_3^2 \right) + \rho x_3 \right) dx$$

Energy conservation (SG)

$$E = \int_{\Omega(t)} \left(\frac{1}{2} \epsilon^2 u_g^2 + \rho x_3 \right) dx$$

Change of variable

$$X = (x_1 + \epsilon u_{g2},\ x_2 - \epsilon u_{g1},\ -\rho)$$
$$w = D_t X$$

Evolution equations

$$w = (u_{g1}, u_{g2}, 0) = \epsilon^{-1} (x_2 - X_2,\ X_1 - x_1,\ 0)$$

Interchange independent/dependent variables, X is independent variable
Define mass in X coordinates as σ, conservation of mass is

$$\partial_t \sigma + \nabla_X \cdot (\sigma w) = 0 \quad \text{on} \quad \Lambda \subset \mathbb{R}^3$$

To solve, determine a map $s : \Lambda \to \Omega(t)$
Choose $s(t,\cdot)$ to minimise E_g for a given $\sigma(t,\cdot)$ under the constraint of mass conservation, so that

$$s(t,\cdot)_\# \sigma(t,\cdot) = \nu_h \equiv \chi_{\Omega_2 \times [0, h(t,x_1,x_2)]}.$$

Write the associated energy as $E_\sigma(h)$. h is determined by the energy minimisation. Can show that this implies

$$(u_{g2}, -u_{g1}, -\rho) = \nabla p \text{ for some } p$$

As with other SG results, we expect to show that the solution is characterised by $\nabla p \in W^{1,\infty}$ (will not prove this)
The theory of Ambrosio and Gangbo is used to solve the evolution equation if w is BV and divergence free (will not prove in this case)

Do only the bits of proof not covered by Benamou and Brenier and Cullen and Gangbo.

Consider the admissible h. Define $\Gamma_t \subset \Omega_2 \times [0,\infty)$ as the subset occupied by fluid at time t. Mass conservation implies that Γ_t has the same volume as Γ_0, defined to be 1. Let $\bar{h}$ be the multivalued function s.t.

$$(x_1, x_2, \bar{h}(t, x_1, x_2)) = \partial \Gamma_t$$

Define $h(t, x_1, x_2) = \int_0^\infty \chi_{\Gamma_t}(x_1, x_2, x_3)\, dx_3$.

Let $v_h(t,x) \in \mathcal{P}_{ac}(\mathbb{R}^3) = \chi_{\Omega_2 \times [0, h(t,\cdot)]}$

$$v_{\bar{h}}(t,x) = \chi_{\Gamma_t}$$

Claim that $E_\sigma(\bar{h}) \geq E_\sigma(h)$

Proof

Let R be a transport map $R_\# v_{\bar{h}} = v_h$ with inverse R^{-1}

Can write $R^{-1} = (R_1^{-1}(x), R_2^{-1}(x), R_3^{-1}(x)) = (x_1, x_2, x_3 + \phi(x_1, x_2, x_3))$ with $\phi \geq 0$

Let $\sigma \in \mathcal{P}_{ac}(\Lambda)$ and S be the optimal map $\Lambda \to \Gamma_t$ with inverse T s.t. $T_\# v_{\bar{h}} = \sigma$.

Since $X_3 < 0$ and $T \circ R^{-1}{}_\# v_h = \sigma$

$$E_\sigma(\bar{h}) = \int_{\Gamma_t} c(x, Tx)\, v_{\bar{h}}\, dx$$

where $c(x, Tx) = \frac{1}{2}((x_1 - Tx_1)^2 + (x_2 - Tx_2)^2) - x_3 \cdot Tx_3$

$$= \int_{\Gamma_t} c(R^{-1}(x), T R^{-1}(x))\, v_h\, dx$$

$$\geq \int_{\Gamma_t} c(x, TR^{-1}(x))\, v_h\, dx \quad \text{as } c \text{ has term } -x_3 TR^{-1}_3 x$$

$$\geq \inf \int_{\Gamma_t} c(x, \hat{T}(x))\, v_h\, dx \quad \text{where } \hat{T}_\# v_h = \sigma$$

$$= E_\sigma(h)$$

Now can assume h is single-valued. Consider regularity,

Since $\frac{\partial p}{\partial x_3} = -\rho$

$$p = p_h \quad \text{at} \quad x_3 = h(t, x_1 x_2)$$

Define $\tilde{p} = p(x_1, x_2, h(x_1 x_2))$, then

$$\nabla_{\{1,2\}} \tilde{p} = \nabla_{\{1,2\}} p + \frac{\partial p}{\partial x_3} \nabla_{\{1,2\}} h = 0$$

so $\nabla_{\{1,2\}} p = \rho \nabla_{\{1,2\}} h$

Since ρ is bounded by its initial values,

$$p \in W^{1,\infty} \Rightarrow h \in W^{1,\infty}$$

Therefore appropriate to minimize $E_\sigma(h)$ in the class

$$H_0 = \{ h : [0,\tau) \times \Omega_2 \to [0,\infty),\ h(t,.) \in W^{1,\infty}(\Omega_2),\ \|h(t,.)\| = 1 \}$$

$$\nu_h = \chi_{\Omega_2 \times [0, h(t,.)]}$$

Show that $E_\sigma(h)$ is convex in h

$$E_\sigma(h) = \inf_{T \# \nu_h = \sigma} \Big[\underbrace{\int_{\Omega_2} \int_0^h \frac{\epsilon^2}{2} \big((x_1 - T_1 x)^2 + (x_2 - T_2 x)^2 \big) dx}_{E_1} + \underbrace{\int_{\Omega_2} \int_0^h -x_3 T_3 x \, dx}_{E_2} \Big]$$

$h \to E_1(h)$ convex:

Set $h_\lambda = (1-\lambda) h_0 + \lambda h_1 \in H_0$

$\nu_0, \nu_1, \nu_\lambda = \nu_{h_0}, \nu_{h_1}, \nu_{h_\lambda}$

$\exists$ maps Φ_0, Φ_1 with

$\Phi_i = (x_1, x_2, \phi_i(x))$ $i = 0, 1$ such that

$\Phi_0 \# \nu_\lambda = \nu_0$ $\quad \Phi_1 \# \nu_\lambda = \nu_1$

~~and optimal maps~~ $\cancel{T_0 T_1 \text{ s.t. } (T_0 \Phi_0) \# \nu_\lambda = \sigma}$,

$\cancel{(T_1 \Phi_1)_\# \nu_\lambda = \sigma}$ and optimal maps T_0, T_1: $T_{0\#} \nu_0 = \sigma$, $T_{1\#} \nu_1 = \sigma$

Then $(T_0 \Phi_0)_\# \nu_\lambda = \sigma$, $(T_1 \Phi_1)_\# \nu_\lambda = \sigma$

. Write $x = (x_1 x_2)$

Then $E_1(h_\lambda) = \inf_{T\# v_\lambda = \sigma} \int \frac{\epsilon^2}{2}(x - Tx)^2 \; v_\lambda \, dx\, dx_3$

$= (1-\lambda) \inf_{T\# v_\lambda = \sigma} \int \frac{\epsilon^2}{2}(x - Tx)^2 \; v_\lambda \, dx\, dx_3$

$+ \lambda \inf_{T\# v_\lambda = \sigma} \int \frac{\epsilon^2}{2}(x - Tx)^2 \; v_\lambda \, dx\, dx_3$

$\leq (1-\lambda) \int \frac{\epsilon^2}{2}(x - T_0 \Phi_0 x)^2 \; v_\lambda \, dx\, dx_3$

$+ \lambda \int \frac{\epsilon^2}{2}(x - T_1 \Phi_1 x)^2 \; v_\lambda \, dx\, dx_3$

$\leq (1-\lambda) \int \frac{\epsilon^2}{2}(x - T_0 x)^2 \; v_0 \, dx\, dx_3$

$+ \lambda \int \frac{\epsilon^2}{2}(x - T_1 x)^2 \; v_1 \, dx\, dx_3$

$h \to E_2(h)$ strictly convex

Set $\tilde{x}_3 = x_3/h$ if $h > 0$

$= 0$ if $h = 0$

$E_2(h) = \inf_{T\# v_h = \sigma} \int_{\Omega_h} \left(\int_0^1 \tilde{x}_3 \, \rho(x_1, x_2, h\tilde{x}_3) \, d\tilde{x}_3 \right) h^2 \, dx_1 \, dx_2$

Since $\rho = -\frac{\partial p}{\partial x_3} = -\frac{1}{h}\frac{\partial p}{\partial \tilde{x}_3}$; ϕ

first integral is $\left[-\int_0^1 \frac{1}{h} \tilde{x}_3 \, dp \right] = \cancel{\left[\frac{1}{h} p \tilde{x}_3\right]_0^1} + \int_0^1 \frac{1}{h} p \, dx_3$

~~Since p decreases with x_3~~ $= -\frac{1}{h} p(1) + \int_0^1 \frac{1}{h} p \, dx_3$

Since p decreases with x_3, this is strictly positive unless $h = 0$

So $E_2(h)$ strictly convex.