Equivalent boundary conditions for Monge-Ampère solutions of the L2 Optimal Transportation problem

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http://arxiv.org/pdf/1208.4870.pdf

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L^2 Optimal Transportation

- Source/Target Data : $d\rho_X(x)(= \rho_X(x)\, dx),\ d\rho_Y(y)(= \rho_Y(y)\, dy)$ s.t. $X, Y \subset R^2$, $\rho_{X,Y} > 0$, $\int_X \rho_X(x)dx = \int_Y \rho_Y(y)dy$, X, Y convex.

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- Boundary conditions are replaced by state constraints :

$$\nabla\psi(\overline{X}) \subset \overline{Y}.$$

  This is called the "Second Boundary Value problem" (Classical solution studied in Delanoë, Urbas ..) .

OUR GOAL :

Solve with a PDE discretization method

$$(MA) \qquad < u > + \, det(D^2 u(x))\rho_Y(\nabla u(x)) = \rho_X(x), \;\; on \; X$$

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- Brenier/Aleksandrov solutions are too weak. Give a meaning to solutions of (MA-HJ) for which discretizations with "good" (convergence, fast solver) properties are available $\rightarrow$ Viscosity solutions setting in section C.

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- ”Face to Face” Neuman type BCs : ex. square to square : $u_{x_1}(\pm 1, .) = \pm 1, \quad u_{x_2}(., \pm 1) = \pm 1$.

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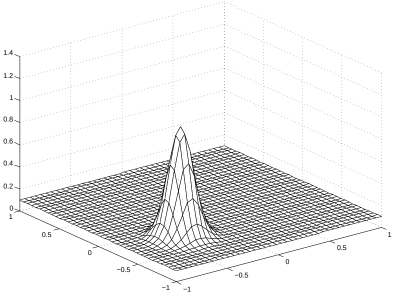
- BUT it assumes a priori knowledge on the boundary to boundary map : illustration on a square to rhombus map.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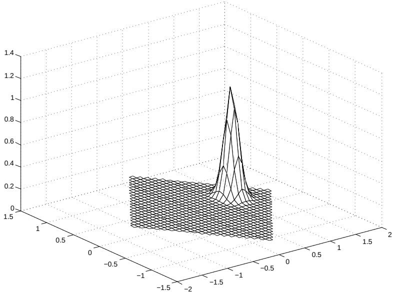
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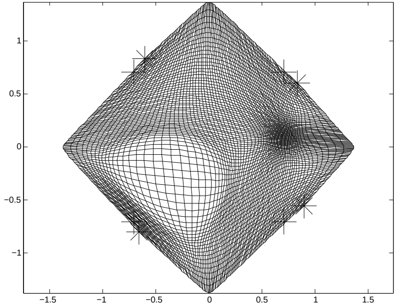
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Equivalent Hamilton-Jacobi equation on the boundary

- Use the ("Y defining function" ... Delanoë, Urbas : Let $H(y)$ convex such that :

$$\begin{cases} H(y) < 0, & y \in Y \\ H(y) = 0, & y \in \partial Y \\ H(y) > 0, & y \in Y^c \end{cases}$$

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- Then (use convexity) $(BV2) \Leftrightarrow (HJ)\ :\ H(\nabla u(x)) = 0,\ x \in \partial X.$
- We use the signed Euclidean distance to ∂Y :

$$H(y) = \begin{cases} +\,\mathrm{dist}(y, \partial Y), & y \in \overline{Y}, \\ -\,\mathrm{dist}(y, \partial Y), & y \in Y^c. \end{cases}$$

H properties and usage

- Obliqueness : Let $y = \nabla u(x)$, $x, y \in \partial X, \partial Y$ and n_x, n_y then exterior normals at x, y. Using $H(\nabla u) = 0$ on ∂X (by definition) and $n_y = \nabla H(y)$ (by construction) we get

$$(OBL) \qquad n_x \cdot n_y \geq 0$$

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- Dual formulation of H : using the supporting hyperplane theorem

$$\begin{cases} H(y) = \sup_{\|n\|=1}\{n \cdot y - H^*(n))\} \\ H^*(n) = \sup_{y_0 \in \partial Y}\{n \cdot y_0\} \end{cases}$$

- On the boundary the sup is attained for n_y the exterior normal to Y at $y = \nabla u(x)$
  (Not known a priori) :

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&\approx \sup_{\{\|n\|=1,\, n \cdot n_x > 0\}}\{\frac{u(x) - u(x - n\,h)}{h} - H^*(n)\}.
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Using the discontinuous viscosity solution framework

(Lions, Perthame, Barles, Souganidis ...)

Theorem : Assuming X, Y convex, $\rho_{X,Y} > 0$ and ρ_Y Lipschitz then $(MA - HJ)$ has a unique convex viscosity solution (using results of Barles, Perthame, Souganidis ...).

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REMARKS :

- The theorem holds for $\rho_X \geq 0$ ((MA) is degenerate elliptic). The solution is the convex envelope extension of the optimal transportation solution which is well defined only on $supp(\rho_X) > 0$.
- (OBL) is an important ingredient to get uniqueness (Non-Linear Neuman BCs, Barles, 1993).

Discretization and Solver

$$MA(u) = \begin{cases} det(D^2 u) - \dfrac{\rho_X}{\rho_Y(\nabla u)} - < u >, & \text{sur } X \\ \sup_{\{\|n\|=1,\, n \cdot n_x > 0\}} \{\nabla u \cdot n - H^*(n)\}, & \text{sur } \partial X \end{cases}$$

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- : –(Accuracy is limited by the stencil width which cannot be arbitrarily increased in practice. (Hybrid FD/Wide Stencil discretization partly fix this problem).

Reconstruction of geodesics $x \mapsto (1 - \frac{t}{T})x + \frac{t}{T}\nabla u(x), \;\; t \in]0, T[$

Ellipse to Ellipse, constant density densité

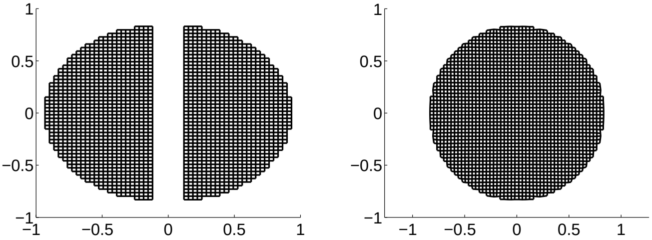
N_X	Max. error N_Y 32	64	128	256	512	Iterations	Time (s)
32	0.0691	0.0667	0.0662	0.0660	0.0660	4	0.2
64	0.0306	0.0284	0.0279	0.0277	0.0277	4	0.5
128	0.0203	0.0176	0.0169	0.0167	0.0167	4	1.7
256	0.0127	0.0096	0.0088	0.0086	0.0088	5	10.1
512	0.0086	0.0056	0.0047	0.0045	0.0047	5	52.2

Table: Exact gradient error $\#$ Newton itération de Newton, timing for $N_y = 512$. Wide-Stencil : 9pts

Illustration 2D ($x \mapsto (1 - \frac{t}{T})x + \frac{t}{T}\nabla\phi^*(x),\ \ t \in]0, T[$)

Discussion on Caffarelli counter example :

Constant density - one non convex domain



$\leftarrow$: Brenier (not Aleksandrov) Solution.

$\rightarrow$: Viscosity solution (non strictly convex) de viscosité. We can do it :-)

Inverse Caffarelli example

N_X	Max. Error N_Y 32	64	128	256	512	Iterations	Time (s)
32	0.0280	0.0284	0.0286	0.0286	0.0286	3	0.2
64	0.0158	0.0164	0.0165	0.0165	0.0165	3	0.4
128	0.0092	0.0093	0.0092	0.0092	0.0092	3	1.3
256	0.0047	0.0036	0.0036	0.0036	0.0036	4	8.3
512	0.0049	0.0040	0.0034	0.0033	0.0033	5	51.7

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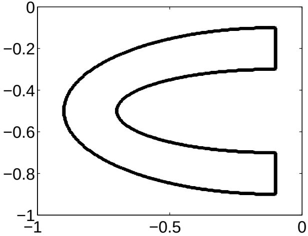
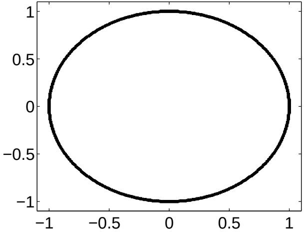
Non convex $\rho_X > 0$ support





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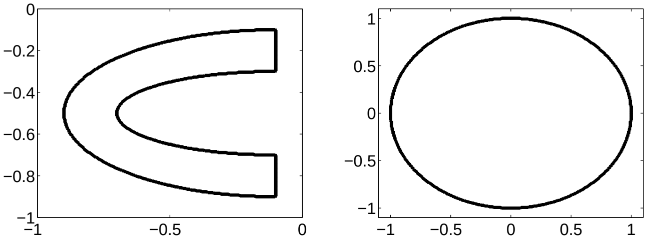
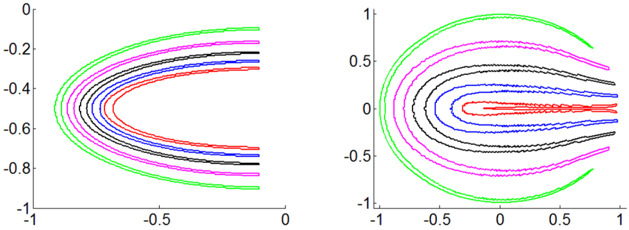
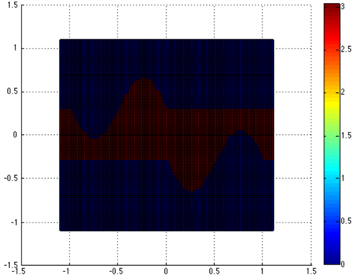


Markers displacements



Toy S.-G. case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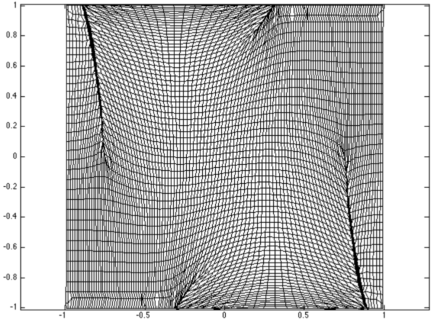
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THANK YOU for your attention.

https://team.inria.fr/mokaplan/