Demixing of viscous fluids

1) Brenier-Benamou

$$\inf\left\{\int_0^1\int \underset{\geq 0}{m}|u|^2\,dx\,dt \;\middle|\; \partial_t m+\nabla\cdot(mu)=0\right\} \quad \text{convex}$$

$$= W_2^2(m_1,m_0) := \inf_{\substack{\pi \to m_0\\ \to m_1}} \int |x_1-x_0|^2\,\pi(dx_1 dx_0)$$

E-L: pressureless irrotational Euler

2) Arnol'd

$$\inf\left\{\int_0^1\int |u|^2\,dx\,dt \;\middle|\; \begin{array}{l}\partial_t m+\nabla\cdot(mu)=0\\ \nabla\cdot u =0\end{array}\right\} \quad \text{kinetic energy}$$

E-L: incompressible Euler

$m \in [0,1]$

$$= \inf\left\{\int_0^1\int m|u|^2\,dx\,dt + \int_0^1\int (1-m)|u|^2\,dx\,dt \;\middle|\; \partial_t m+\nabla\cdot(mu)=0,\ \partial_t(1-m)+\nabla\cdot((1-m)u)=0\right\}$$

$$\underset{\neq}{\geq}\ W_2^2(m_1,m_0)+W_2^2(1-m_1,1-m_0)$$

can but be relaxed to explicit convex

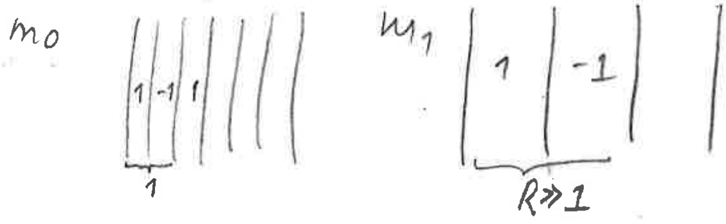
3)

$$\inf\left\{\int_0^1 \underbrace{\int \left|\tfrac12(\nabla+\nabla^t)u\right|^2 dx}\,dt \;\middle|\; \begin{array}{l}\partial_t m+\nabla\cdot(mu)=0\\ \nabla\cdot u =0\end{array}\right\} \quad \text{viscous dissipation}$$

$= \frac12\int|\nabla u|^2dx$ on torus of size L

non-convex, minimizer ✓

Conjecture by Bressan: (Doering et al)



$$\Longrightarrow \int_0^1 \frac{1}{L}\int |\nabla u|\,dx\,dt \gtrsim \log R$$

$$\int_0^1 \sup_x |\nabla u|\,dt \gtrsim \log R \ ✓$$

Theorem of Crippa - De Lellis

$$\overset{1}{\Longrightarrow} \quad \int_0^1 \frac{1}{L^n}\int |M \nabla u| \, dx\, dt \gtrsim \log R$$

$$\overset{p>1}{\Longrightarrow} \quad \left(\int_0^1 \frac{1}{L^n}\int |\nabla u|^p dx\, dt\right)^{1/p} \gtrsim \log R$$

Corollary of Brenier-Otto for $u \in [-1,1]$, $\int u\, dx = 0$

$$\left(\int_0^1 \frac{1}{L^n}\int |\nabla u|^2 dx\, dt\right)^{1/2} \gtrsim \left|\frac{1}{L^n}D(u_1) - \frac{1}{L^n}D(u_0)\right|$$

where

$$D(u) := \inf_{\pi \rightrightarrows \begin{array}{l} u_+ = \max\{u, 0\} \\ u_- = \max\{-u, 0\} \end{array}} \int \log|x_1 - x_0| \, \pi(dx_1 dx_0)$$

Engineering community (Doering): $\int u = 0$
Express this in case of $u \in \{-1, 1\}$ in terms of

$\frac{1}{L^n}\int |\nabla u_0|$ = 2 interfacial area density

$\left(\frac{1}{L^n}\int ||\nabla|^{-1} u_0|^2\right)^{1/2}$ electrostatic energy of charge distrib. u0 density

Seis 13+

Inte. $\frac{1}{L^n}\int |\nabla u_0| \exp\left(C_0 \frac{1}{L^n} D(u_0)\right) \gtrsim 1$

Diss.

$$\exp\left(C_0 \frac{1}{L^n} D(u_0)\right) \lesssim \exp\left(C_1 \left(\int_0^1 \frac{1}{L^n}\int |\nabla u|^2 dx\, dt\right)^{1/2}\right) \cdot \exp\left(C_0 \frac{1}{L^n} D(u_1)\right)$$

$$\frac{1}{L^n}\int |\nabla u_0| \underbrace{\exp\left(C_0 \frac{1}{L^n} D(u_1)\right)}_{\text{Jensen } \lesssim \left(\frac{1}{L^n}\int ||\nabla|^{-1} u_1|^2\right)^{1/2}} \gtrsim \exp\left(-C_1 \left(\int \frac{1}{L^n}\int |\nabla u|^2 dx\, dt\right)^{1/2}\right)$$

Why interested? Gradient flows

1) Riemannian structure behind BB

$$g_m(\delta m, \delta m) = \inf\left\{\int m|u|^2 dx \;\middle|\; \delta m + \nabla\cdot(mu) = 0\right\}$$

$$\underset{0}{\vee\!\!|} \qquad = \int m|\nabla\psi|^2 dx \quad \text{where } \delta m - \nabla\cdot(m\nabla\psi) = 0$$

2) Compare with Euclidean structure

$$g_m(\delta m, \delta m) = \inf\left\{\int |j|^2 dx \;/\; \delta m + \nabla\cdot j = 0\right\}$$

$$= \int |\nabla\mu|^2 dx \quad \text{where } \delta m - \Delta\mu = 0$$

$$= \int ||\nabla|^{-1}\delta m|^2 dx = |\delta m|^2_{\dot H^{-1}}.$$

models "diffusion"

3) Riemannian structure behind viscous BOS...

$$g_m(\delta m, \delta m) = \inf \left\{ \int |\nabla u|^2 dx \;\middle|\; \begin{array}{l} \delta m + \nabla\cdot(m\,u) = 0 \\ \nabla\cdot u = 0 \end{array} \right\}$$

models "(viscous) flow"

Gradient flow on $m \in \{-1,1\}$ of $\int |\nabla m| = 2\,\mathcal{H}^{d-1}(\partial\{m=1\})$

w.r.t. 2) "diffusion" Mullins-Sekerka = early
w.r.t. 3) "flow" Siggia's growth = late
} stage in sp. modal decomposition

$$C_1 \frac{1}{L^n} D(m) \geq \ln \frac{1}{C_0 \frac{1}{L^n} E(m)}$$

$$\frac{1}{L^n} E(m) \exp\left(C_1 \frac{1}{L^n} D(m)\right) \geq \frac{1}{C_0}$$

Demixing in viscous fluids:
Connection with OT

Felix Otto

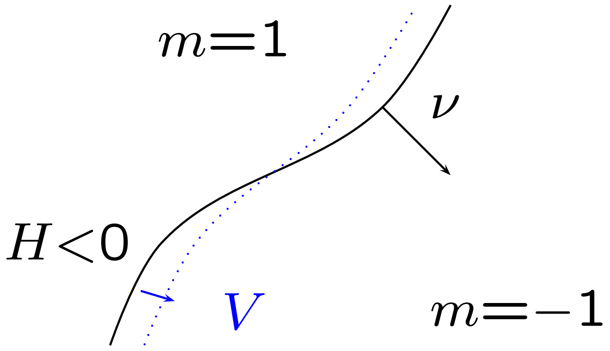
Max Planck Institute for **Mathematics in the Sciences**

Leipzig, Germany

joint work with:

R. V. Kohn, Y. Brenier, C. Seis, D. Slepcev

Geometric evolution equation, diffusion



mean curvature: H

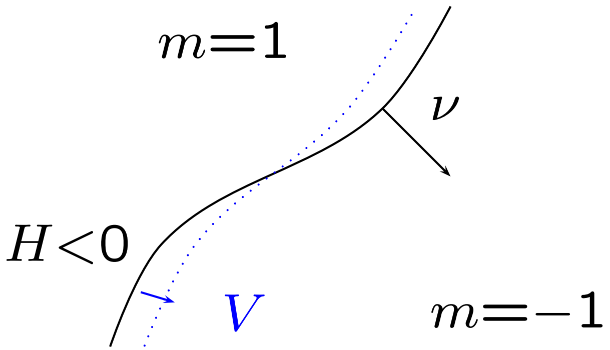
normal velocity: V

$$-\triangle\mu = 0 \quad \text{in bulk,} \quad \left\{ \begin{array}{rcl} \mu & = & \frac{1}{2}H \\ V & = & [\nu\cdot\nabla\mu] \end{array} \right\} \quad \text{on interface}$$

"Mullins-Sekerka" ; Pego, Alikakos&Bates&Chen, Röger & Schätzle

Third-order free boundary problem

Geometric evolution equation, flow



$$\left\{ \begin{array}{rcl} \nabla \cdot u & = & 0 \\ -\nabla \cdot S & = & 0 \end{array} \right\} \text{ in bulk, } \quad \left\{ \begin{array}{rcl} \tau \cdot [S]\nu & = & 0 \\ \nu \cdot [S]\nu & = & -H \\ V & = & \nu \cdot u \end{array} \right\} \text{ on interface,}$$

where $S := \frac{1}{2}(\nabla u + \nabla^t u) - p\,\mathrm{id}$ is stress tensor

"Siggia's growth";

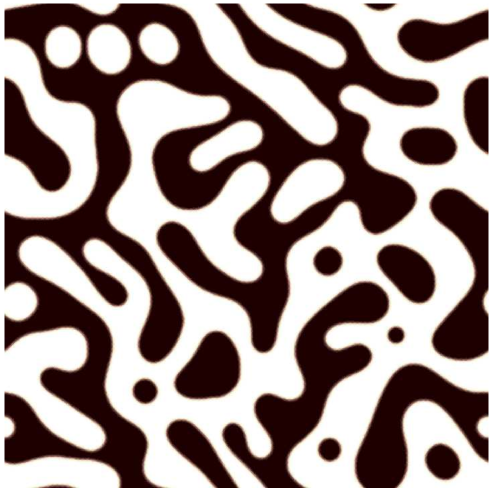
First-order free boundary problem

Experiments/numerics:

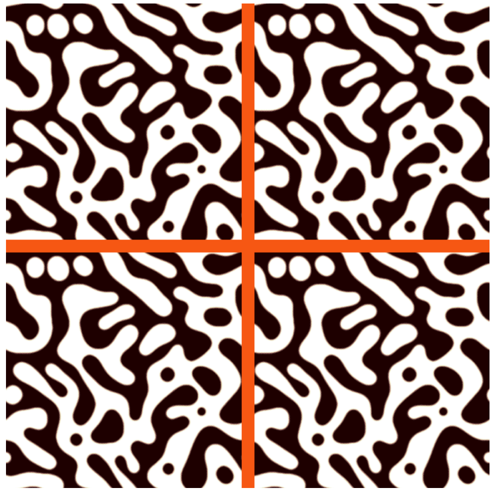
Statistical self-similar coarsening

earlier

later



later,
rescaled,
periodically
extended



Diffusion: coarsening exponent 1/3

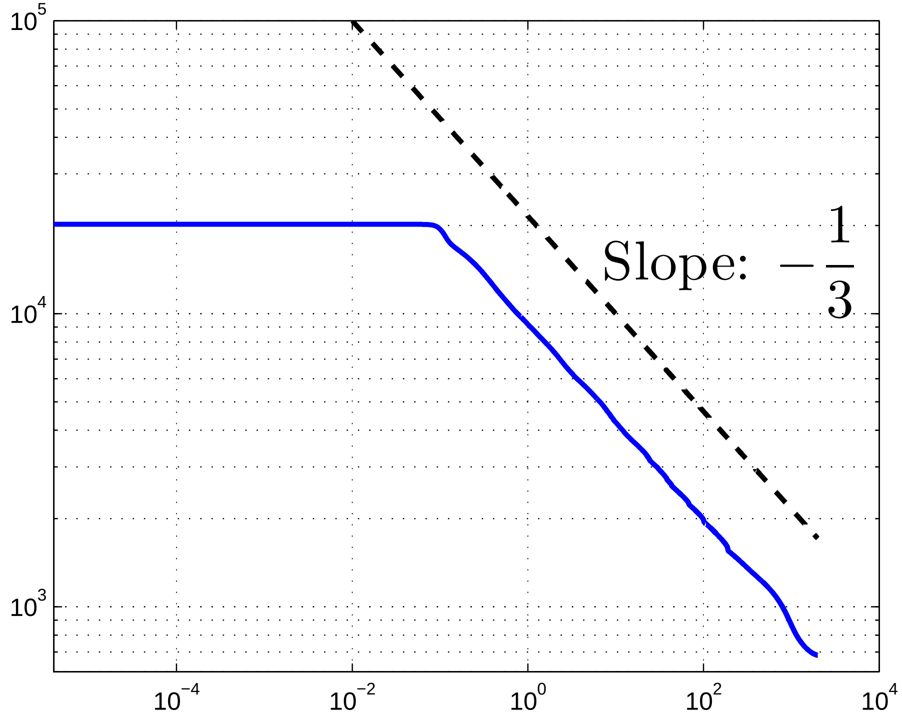
After initial phase: Energy $E(m) \approx$ 2interfacial area

Hence $\frac{1}{L^n}E^{-1}$ is an *average length scale*

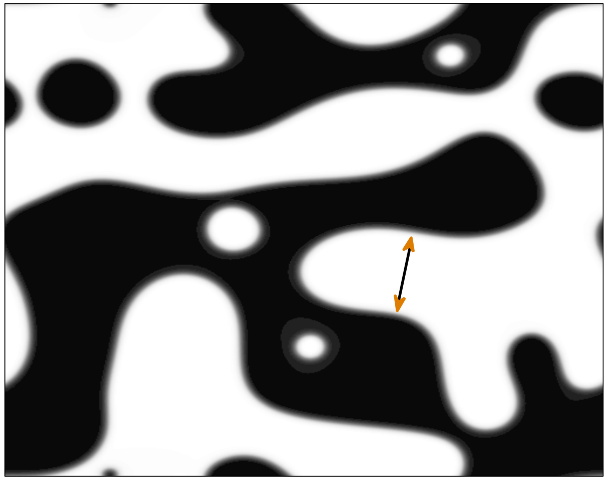
Energy E vs. time t,

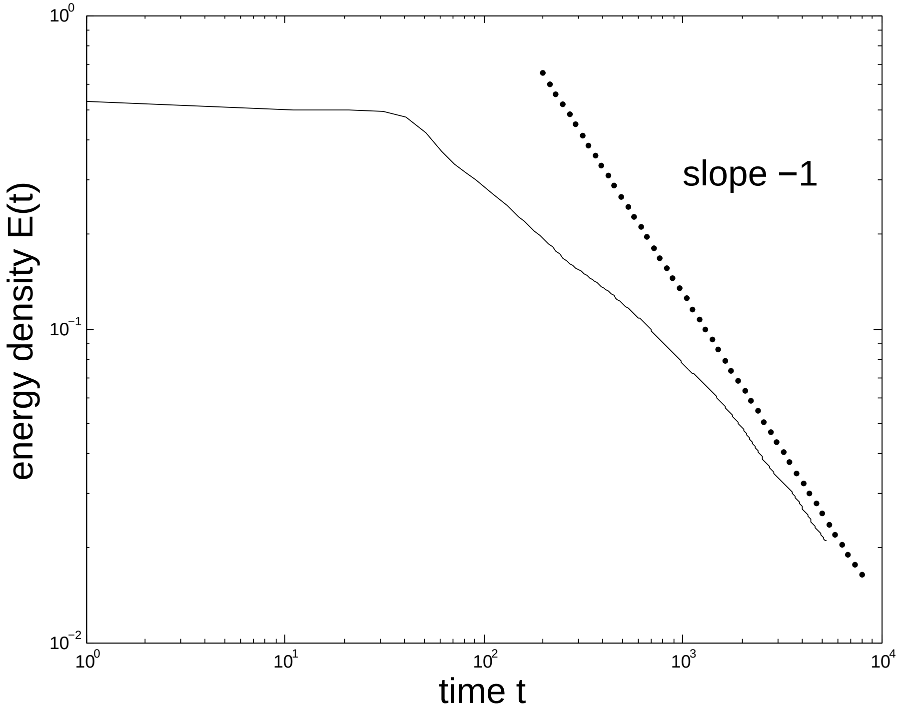
double logarithmic plot:

$\mathbf{L^{-(n=2)}E(m) \sim t^{-1/3}}$



Flow: coarsening exponent 1





$$\mathbf{L^{-(n=2)}\,E(u)\ \sim\ t^{-1}}$$

Value of exponent not a surprise...

Scale invariance (sharp interface level) $\quad \hat{x} = \mu x, \quad \hat{t} = \mu^3 t.$

Hence *if* evolution is statistically self-similar, i. e.

$$|\mathcal{F}m(t,\cdot)(k)|^2 \approx f_{\mathrm{universal}}(t^\gamma k) \quad \text{for } t \gg 1,$$

coarsening exponent γ must be $1/3$:

$$\begin{aligned} f_{\mathrm{universal}}(t^\gamma k) &\approx |\mathcal{F}m(t,\cdot)(k)|^2 \\ &= |\mathcal{F}\hat{m}(\mu^3 t,\cdot)(\mu^{-1}k)|^2 \approx f_{\mathrm{universal}}(\mu^{3\gamma-1} t^\gamma k) \end{aligned}$$

... but statistical self-similarity a mystery

Rigorous analysis of coarsening

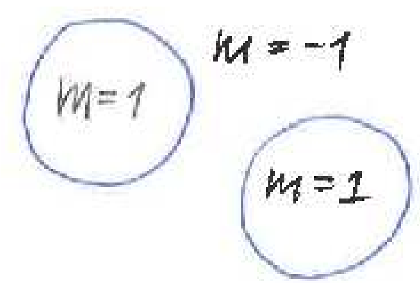
Rigorous treatment

has to cope with ungeneric behavior

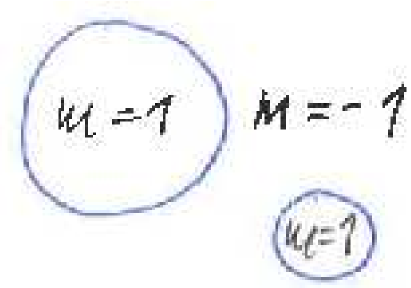
Upper bounds on E not independent of initial data:

— too many stationary points of E

diffusion



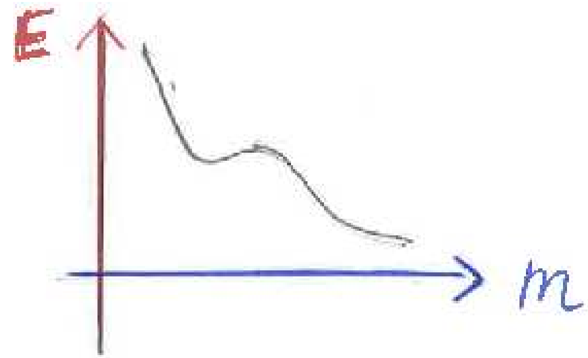
flow



Lower bounds on E independenty of initial data

Basic idea for rigorous lower bounds on E

Dynamics is steepest descent in energy landscape



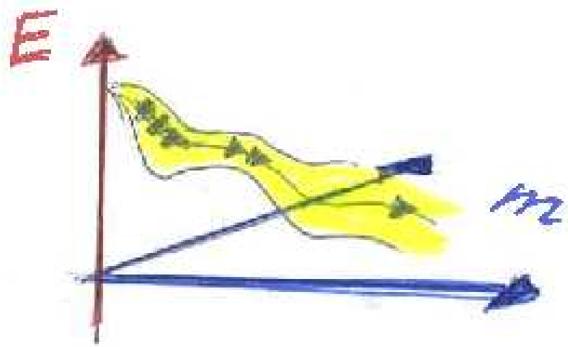
energy $\leftrightarrow$ heights,

dissipation mechanism $\leftrightarrow$ distances

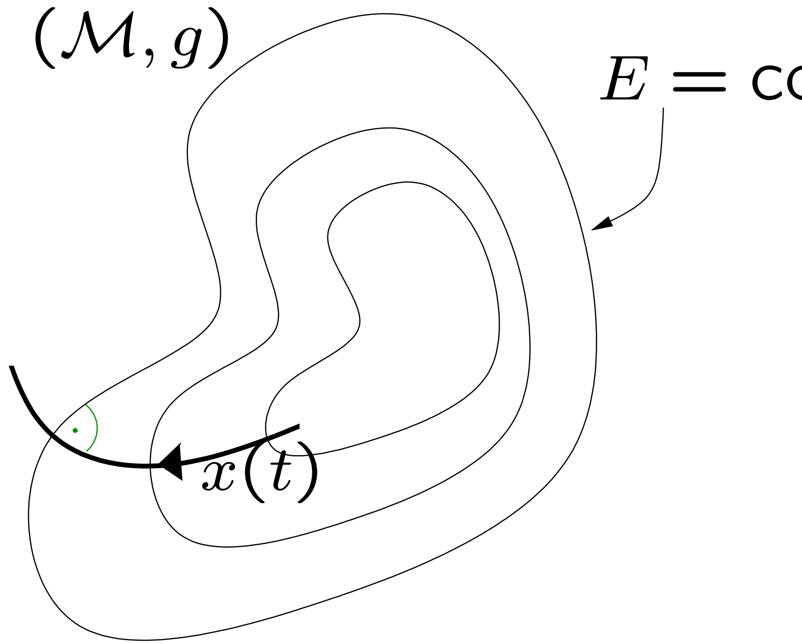
landscape *not steep*

$\Longrightarrow$

energy decreases *not fast*



An abstract framework



$(\mathcal{M}, g)$ Riemannian manifold
E functional on $\mathcal{M}$

Gradient flow $\dot{x} = -\text{grad}_g E(x)$

metric tensor $g_x(\delta x, \delta x)$ $\rightsquigarrow$ induced distance $d(x_0, x_1)$
local *global*

Relating geometry to dynamics

Lemma. (Kohn & O. '02)

Assume for some $\alpha > 0$ and $x^* \in \mathcal{M}$

$$E(x)d(x^*,x)^{\alpha} \gtrsim 1$$

Then for all $\sigma \in (1, \frac{\alpha+2}{\alpha})$

$$\int_0^T E(x(t))^{\sigma}\,dt \gtrsim \int_0^T (t^{-\frac{\alpha}{\alpha+2}})^{\sigma}\,dt \qquad \text{for } T^{\frac{1}{\alpha+2}} \gg d(x^*,x(0))$$

Needed for $\alpha = 1$ (diffusion)

Extension for $\alpha = \infty$ (flow)

Lemma. (Brenier & O. Seis '10)

Assume for some $x^* \in \mathcal{M}$

$$E(x)\,\mathrm{exp}\big(d(x^*, x)\big) \gtrsim 1$$

Then

$$\int_0^T E(x(t))\,dt \gtrsim \int_0^T (t+1)^{-1}\,dt \qquad \text{for } \ln T \gg d(x^*, x(0))$$

Type of dissipation determines metric tensor g

Transport mechanism		type of dissipation:
diffusion	$\rightsquigarrow$	outer friction,
flow	$\rightsquigarrow$	inner friction (viscosity).

$$g_m(\delta m, \delta m) \;=\; \inf\left\{ \int |j|^2\, dx \,\middle|\, \delta m + \nabla\cdot j \;=\; 0 \right\},$$

$$g_m(\delta m, \delta m)$$

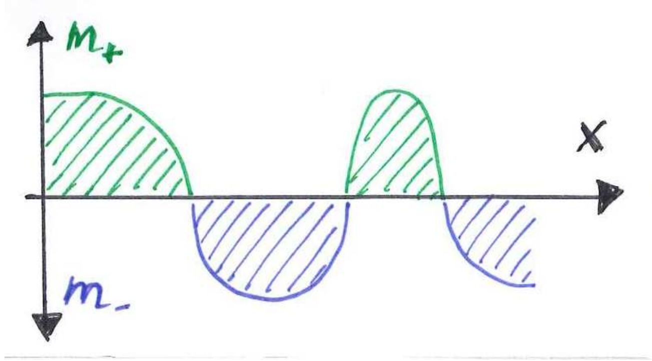
$$=\; \inf\left\{ \int |\nabla u|^2\, dx \,\middle|\, \delta m + \nabla\cdot(m\,u) \;=\; 0,\; \nabla\cdot u \;=\; 0 \right\}.$$

Lower bound D
on induced distance d to reference configuration m^*

Reference configuration: well-mixed state $m^* = 0$

Lower bound D(m) to **induced distance** $d(m, m^*)$
given by **transportation distance**

between $m_+ := \max\{m, 0\}$
and $m_- := \max\{-m, 0\}$

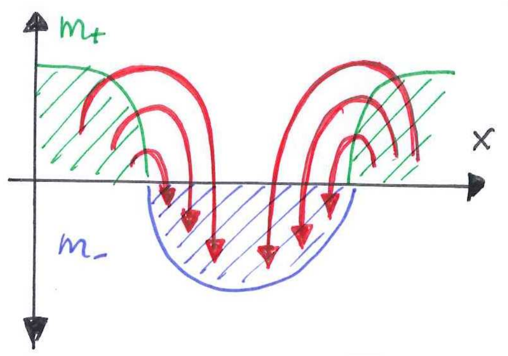


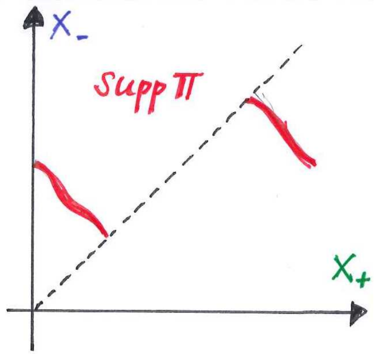
Note that $\int m = 0$ implies $\int m_+ = \int m_-$

Definition of transportation distance

Given $m = m_+ - m_-$, measure $\pi(dx_- \, dx_+)$ on $[0,L]^n \times [0,L]^n$ is called **admissible transfer plan** provided

$$\int \zeta(x_+)\,\pi(dx_+\,dx_-) = \int \zeta(x)\,m_+(x)\,dx,$$
$$\int \zeta(x_-)\,\pi(dx_+\,dx_-) = \int \zeta(x)\,m_-(x)\,dx.$$

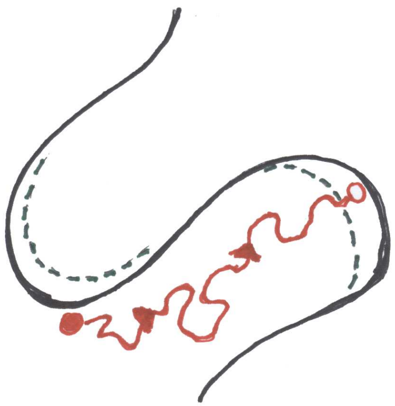


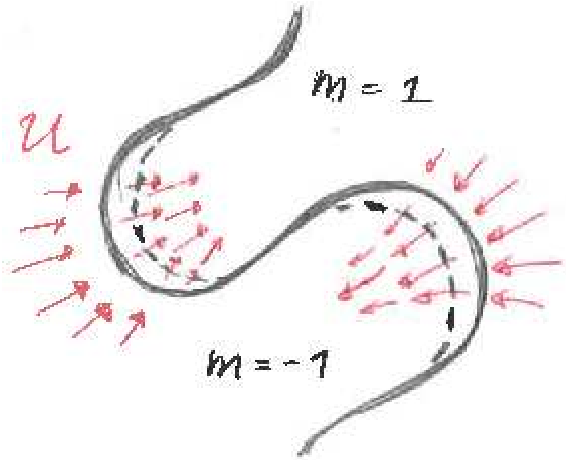


$$\mathbf{D}(m) := \inf_{\pi} \int \mathbf{c}(|x_- - x_+|)\pi(dx_+\,dx_-)$$

Dissipation mechanism determines geometry

Distance on configuration space





diffusion

$\gtrsim$ transportation distance with cost

$c(x_- - x_+)$

$|x_- - x_+|$

flow

$\gtrsim$ transportation distance with cost

$c(x_- - x_+)$

$\ln |x_- - x_+|$

Main result (flow)

Theorem. (O. & Seis & Slepcev '13, Brenier, O. & Seis '11)

For any solution m

$$\int_0^T \frac{1}{L^n} E(m(t))\,dt \gtrsim \int_0^T (t+1)^{-1}\,dt$$

provided

$$\ln T \gg \frac{1}{L^n} D(m(0))$$

Dissipation: $D(m)$ is lower bound to $d(m,m^*)$

Dissipation Lemma (BOS, OSS).

Suppose $\partial_t m + \nabla \cdot (mu) = 0$ and $\nabla \cdot u = 0$.

Suppose $m \in [-1,1]$ and $\int m = 0$.

Then $\frac{d}{dt}\frac{1}{L^n}D(m) \lesssim \left(\frac{1}{L^n}\int |\nabla u|^2 dx\right)^{\frac{1}{2}}$.

Formal consequence: $d(m, m^* = 0) \gtrsim D(m)$

Interpolation: $D(m)$ provides lower bound for $E(m)$

Interpolation Lemma (BOS, OSS, Seis 13+).

There exists $c_0 \in (0, \infty)$ only depending on n such that

$$\frac{1}{L^n}E(m)\,\mathrm{exp}\left(c_0\frac{1}{L^n}D(m)\right) \gtrsim 1.$$

Proof of Dissipation Lemma

Preliminaries

Given flow:
$$\partial_t m + \nabla \cdot (m u) = 0,$$
$$\nabla \cdot u = 0$$

This implies $\partial_t m_\pm + \nabla \cdot (m_\pm u) = 0$

W. l. o. g. u smooth, $t = 0$. Consider flow $\Phi(t, \cdot)$.

Let $\pi^*(dx_+ \, dx_-)$ denote *optimal* transfer plan at $t = 0$

Then $(\Phi(t,\cdot) \otimes \Phi(t,\cdot)) \# \pi^*(dx_+ \, dx_-)$ is *admissible* transfer plan at time t.

Choice of $c = \log$ becames clear

$(\Phi(t,\cdot)\otimes\Phi(t,\cdot))\#\pi^*(dx_+\,dx_-)$ is *admissible* transfer plan at time t (and optimal at $t=0$).

$$D(m(t,\cdot)) \;\leq\; \int c(|\Phi(t,x_-)-\Phi(t,x_+)|)\,\pi^*(dx_-\,dx_+)$$

(and equality at time $t=0$)

$$\frac{d^+}{dt}_{|t=0} D(m(t,\cdot))$$

$$\leq \int \frac{dc}{dz}(|x_- - x_+|)\,\frac{x_- - x_+}{|x_- - x_+|}\cdot(u(0,x_-)-u(0,x_+))\pi^*(dx_-\,dx_+)$$

$$\leq \int \frac{|u(0,x_-)-u(0,x_+)|}{|x_- - x_+|}\,\pi^*(dx_-\,dx_+)$$

Follow Crippa-DeLellis

Have

$$\frac{d^+}{dt}_{|t=0} D(m(t,\cdot)) \le \int \frac{|u(0,x_-) - u(0,x_+)|}{|x_- - x_+|} \pi^*(dx_-\, dx_+)$$

Remains to show for $u = u(0,\cdot)$:

$$\frac{1}{L^n} \int \frac{|u(x_-) - u(x_+)|}{|x_- - x_+|} \pi^*(dx_-\, dx_+) \le \left(\frac{1}{L^n} \int |\nabla u|^2 dx \right)^{1/2}$$

Use idea of Crippa-DeLellis ('08) for quantification of DiPerna-Lions theory on uniqueness for transport equations $\partial_t m + u \cdot \nabla m = 0$

Use maximal function

Recall maximal function:

$$(M\, f)(x) \;:=\; \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} f\, dy.$$

Crucial properties:

$$\frac{|u(x_-) - u(x_+)|}{|x_- - x_+|} \;\lesssim\; (M\, |\nabla u|)(x_-) + (M\, |\nabla u|)(x_+)$$

$$\int |M\, f|^2\, dx \;\lesssim\; \int |f|^2\, dx$$

Conclusion

$$\frac{1}{L^n}\int \frac{|u(x_-) - u(x_+)|}{|x_- - x_+|}\,\pi^*(dx_-\,dx_+)$$

$$\lesssim \frac{1}{L^n}\int \big((M\,|\nabla u|)(x_-) \;+\; (M\,|\nabla u|)(x_+)\big)\;\pi^*(dx_-\,dx_+)$$

$$= \frac{1}{L^n}\int (M\,|\nabla u|)m_-\,dx \;+\; \frac{1}{L^n}\int (M\,|\nabla u|)m_+\,dx$$

$$\lesssim \left(\frac{1}{L^n}\int (M\,|\nabla u|)^2\,dx\right)^{1/2} \sup|m|$$

$$\lesssim \left(\frac{1}{L^n}\int |\nabla u|^2\,dx\right)^{1/2}$$

Interpolation Lemma in context

Case of diffusion

... is gradient flow w. r. t. H^{-1}-metric:

$$\begin{aligned} g_m(\delta m, \delta m) &= \inf \left\{ \int |j|^2\, dx \,\middle|\, \delta m + \nabla \cdot j = 0 \right\} \\ &= \int |\nabla \mu|^2\, dx \quad \text{where } \delta m - \triangle \mu = 0. \end{aligned}$$

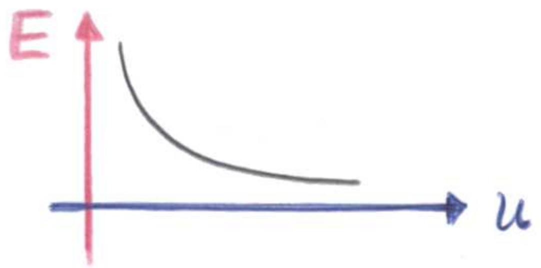
Euclidean case:

infinitesimal metric g = distance function d^2

Interfacial energy: $E(m) := 2\mathcal{H}^{n-1}(\partial\{m = 1\})$

Energy landscape via interpolation estimates

Flatness of energy landscape



$$\underbrace{L^{-n}\mathcal{H}^{n-1}(\partial\{m\approx 1\})}_{\text{Energy}}\ \underbrace{\left(L^{-n}\int ||\nabla|^{-1}m|^2dx\right)^{\frac{1}{2}}}_{\text{distance to } m^*=0} \gtrsim 1 \approx \left(L^{-d}\int |m|^{\frac{4}{3}}dx\right)$$

$$\rightsquigarrow \qquad \|\nabla m\|_{L^1}^{\frac{1}{2}}\ \||\nabla|^{-1}m\|_{L^2}^{\frac{1}{2}} \gtrsim \|m\|_{L^{\frac{4}{3}}}$$

Cohen & Dahmen & Daubechies & DeVore, Ledoux

An improvement for $n = 2$

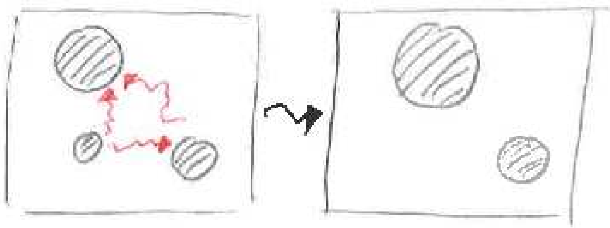
Lemma (Cinti & O. '13+, Viehmann & O. '09).

Suppose $n = 2$ and $m \geq -1$. Then

$$\|m\|_{L^{4/3}\ln^{1/4} L} := \left(\int_{m\geq 1} m^{4/3} \ln^{1/3} m \, dx\right)^{3/4} \lesssim \|\nabla m\|_{L^1}^{\frac{1}{2}} \, \||\nabla|^{-1} m\|_{L^2}^{\frac{1}{2}}.$$

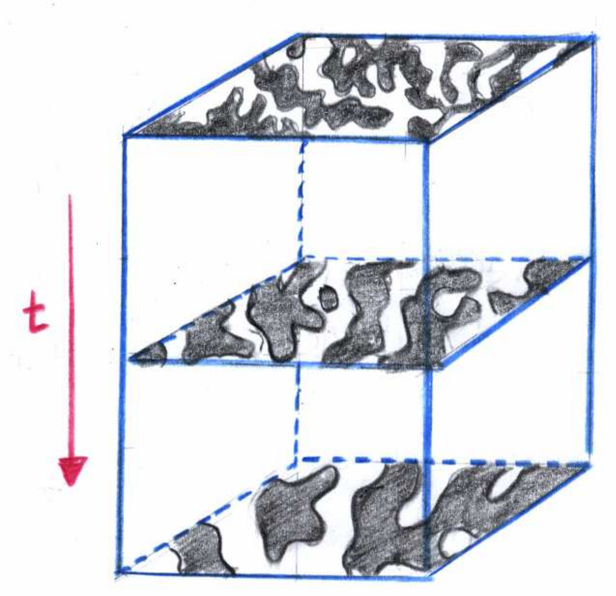
Useful for Ostwald ripening

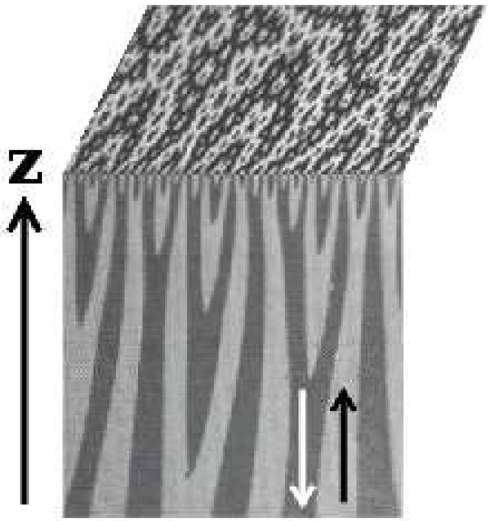
(Conti & Niethammer & O. '06)



Different physics — same mathematics

Domain branching in ferromagnets



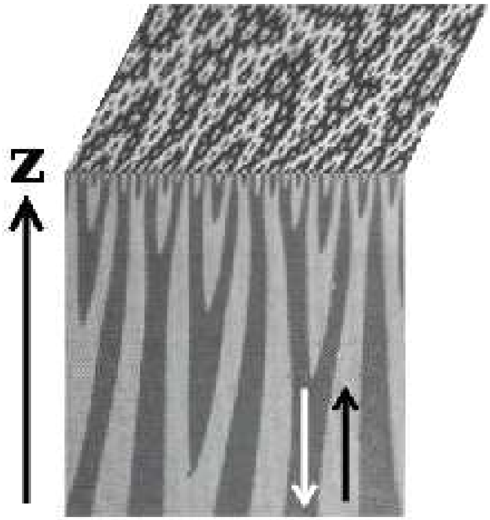


Same interpolation estimate:

$$\|m\|_{L^{\frac{4}{3}}} \quad \lesssim \quad \|\nabla m\|_{L^1}^{\frac{1}{2}} \quad \||\nabla|^{-1} m\|_{L^2}^{\frac{1}{2}}$$

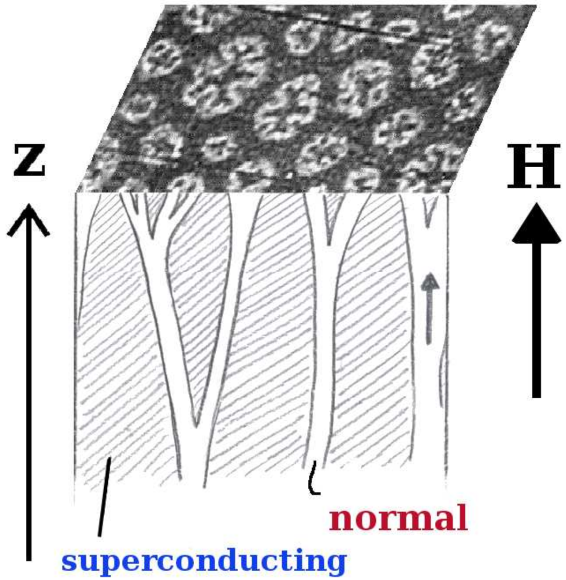
A universal pattern

Domain branching in ferromagnets



Hubert,
Choksi & Kohn

Domain branching in superconductors



Landau,

Choksi & Kohn & O.

Twin-splitting in shape memory alloys

Kohn & Müller,
Conti

Future directions

Local estimates

Example with cross-over due to dissipation mechanisms

“in series”, like diffusion+attachment-limited

— instead of “in parallel”, like diffusion+flow-mediated

Future directions

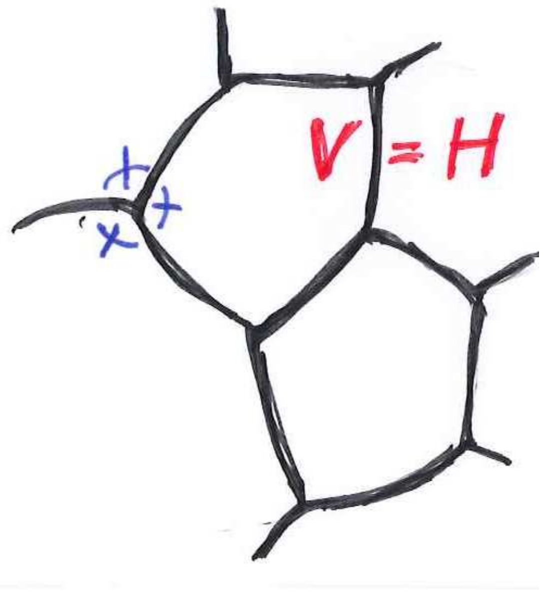
Grain growth

= aging in polycrystals

= multi-component

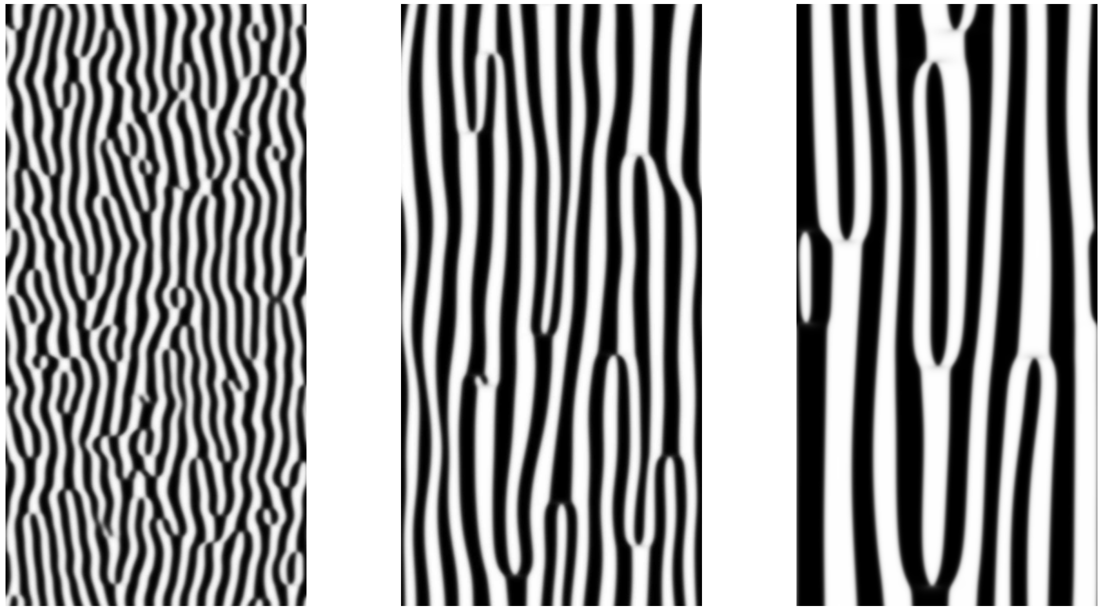
mean curvature flow

$\frac{1}{L^n} E \gtrsim t^{-1/2}$



Future directions

Defect-mediated coarsening (e.g. in Siegert's model for crystal growth)



Upper bounds on E for *generic* initial data