Gevrey spaces and nonlinear inviscid damping for 2D Euler.

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joint work with
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First part is joint with Tak Kwong Wong (UPenn) and David Gérard-Varet (Paris 7)
Third part joint is with J. Bedrossian and Clement Mouhot

Gevrey classes. Maurice Gevrey (1884-1957)

In 1918, Maurice Gevrey defined the following:

Definition: Let $m \geq 1$. $G^m(\mathbb{T})$ (*Gevrey space of class m*) is the set of $f = f(x)$ s.t.

$$\exists C, \tau > 0, \quad |f^{(k)}(x)| \leq C\,\tau^{-k}(k!)^m, \quad C, \tau > 0, \quad \forall\, k,\, x.$$

Remark:

- $m = 1$: analytic functions.
- $m > 1$: $G^m(\mathbb{T})$ contains compactly supported functions.

Proposition: $f \in G^m(\mathbb{T})$ iff

$$\exists C, \sigma > 0, \quad |\hat{f}(k)| \leq Ce^{-\sigma k^{1/m}}$$

Gevrey norms

- In physical space

$$\|f\|^2_{G^{m,\sigma}_\tau} := \sum_{j\in N} \left(\tau^j (j!)^{-m} j^\sigma\right)^2 \|\partial^j f\|^2_{L^2} \tag{1}$$

- In Fourier space

$$\|f\|^2_{\mathcal{G}^{m,\sigma}_\tau} := \||\xi|^\sigma e^{\tau|\xi|^{1/m}} \hat{f}(\xi)\|^2_{L^2} \tag{2}$$

m is the Gevrey class
σ is a Sobolev correction
τ is the *radius* of analyticity when $m = 1$.

Gevrey norms (mostly for analytic regularity) are used in many PDE problems :

- Temam-Foias
- Bardos-BenAchour
- Ferrari-Titi
- Levermore-Oliver-Titi
- Sammartino and Caflisch
- Kukavica-Temam-Vicol-Ziane
- Rauch

Prandtl system: Ludwig Prandtl (1875 -1953)

Aerodynamic boundary layers were first introduced by Ludwig Prandtl in a paper presented on August 12, 1904 at the third International Congress of Mathematicians in Heidelberg, Germany. It divided the flow field into two areas:

1) One inside the boundary layer, dominated by viscosity and creating the majority of drag experienced by the boundary body (Prandtl system),

2) One outside the boundary layer, where viscosity can be neglected without significant effects on the solution (Euler)

D'Alembert's paradox. D'Alembert (1717-1783)

In particular, this gave an answer to the D'Alembert's (1717-1783) paradox (or the hydrodynamic paradox). This paradox was a contradiction reached in 1752 by French mathematician Jean le Rond D'Alembert who proved that for incompressible and inviscid potential flow the drag force is zero on a body moving with constant velocity relative to the fluid.

Starting point: Navier-Stokes in a half-plane ($\Omega = \mathbb{R}^2_+$).

Dimensionless form:

$$\boxed{\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \varepsilon\Delta\mathbf{u} = 0, \\ \nabla\cdot\mathbf{u} = 0. \end{cases}} \tag{NS}$$

$\varepsilon = \frac{\nu}{UL}$ (ε^{-1} is the *Reynolds number*)

Flow around an airplane wing: $\varepsilon \sim 10^{-5}$-10^{-6}.

Tempting simplification : $\varepsilon = 0$.

$$\boxed{\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = 0, \\ \nabla\cdot\mathbf{u} = 0. \end{cases}} \tag{E}$$

Case $\Omega = \mathbb{R}^2$: this simplification can be justified *in general* in the case without boundaries (Swann, Kato, Constantin, ...).

Case $\Omega = \mathbb{R}^2_+$: not so clear !

Problem with the boundary condition.

- If $\varepsilon = 0$ (Euler): *non-penetration* condition:

$$\boxed{\mathbf{u} \cdot n|_{\partial\Omega} \ = \ 0}$$

- If $0 < \varepsilon \ll 1$ (Navier-Stokes): *no-slip* condition:

$$\boxed{\mathbf{u}|_{\partial\Omega} \ = \ 0}$$

Strong velocity gradients, near the boundary: *boundary layer*.

Question: How does concentration in the boundary layer affect the asymptotics $\varepsilon \to 0$?

Prandtl boundary layer theory (1904)

Asymptotic model, involving two different asymptotic expansions of the solution $\mathbf{u}^\varepsilon$:

- away from the boundary: Euler:

$$\boxed{\mathbf{u}^\varepsilon \approx \mathbf{u}_E = (u_E, v_E)(t,x,y).}$$

- near the boundary: concentration at scale $\sqrt{\varepsilon}$:

$$\boxed{\mathbf{u}^\varepsilon \approx \left(u\left(t,x,y/\sqrt{\varepsilon}\right), \sqrt{\varepsilon}\, v\left(t,x,y/\sqrt{\varepsilon}\right)\right)}$$

where $u = u(t,x,Y)$, $v = v(t,x,Y)$, $(x,Y) \in \mathbb{R} \times [0,+\infty[$.

Formally (with y instead of Y):

$$\boxed{\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p - \partial_y^2 u = 0, & x \in \mathbb{R},\ y > 0, \\ \partial_y p = 0, & x \in \mathbb{R},\ y > 0, \\ \partial_x u + \partial_y v = 0, & x \in \mathbb{R},\ y > 0. \end{cases}}$$

Boundary conditions:

- no-slip: $u(t,x,0) = v(t,x,0) = 0.$
- matching to the Euler flow:

$$\begin{aligned} u(t,x,+\infty) &= U_E(t,x) := u_E(t,x,0), \\ p(t,x,+\infty) &= P_E(t,x) := p_E(t,x,0). \end{aligned}$$

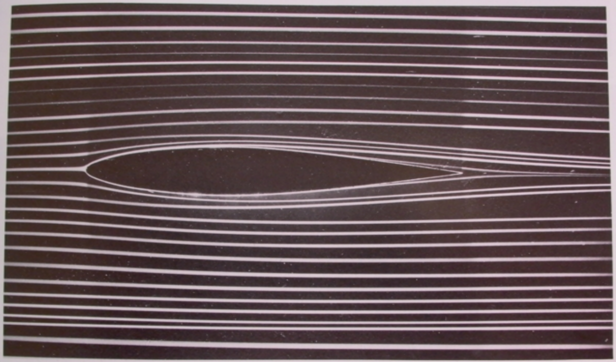
Finally, Prandtl equation reads (with $(x,y) \in \mathbb{R}^2_+$):

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = -\partial_x P_E, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \\ u|_{y=+\infty} = U_E. \end{cases} \qquad \text{(P)}$$

Questions: Experimental evidence ? Mathematical justification ?

More precisely: is (P) well-posed ? Asymptotic expansion of $\mathbf{u}^\varepsilon$?

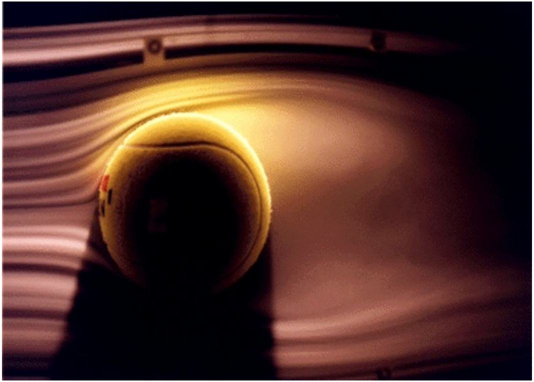
Many experimental studies (flows around obstacles)...



23. Symmetric plane flow past an airfoil. An NACA 64A015 profile is at zero incidence in a water tunnel. The Reynolds number is 7000 based on the chordlength. Streamlines are shown by colored fluid introduced upstream. The flow is evidently laminar and appears to be unseparated, though one might anticipate a small separated region near the trailing edge. *ONERA photograph, Werlé 1974*

... which exhibit many instabilities.

<u>Example</u>: Boundary layer separation.



Explanation for the separation: *adverse pressure gradient*.

$U_E > 0$, $-\partial_x P_E < 0$. Loss of monotonicity (in y), followed by separation.

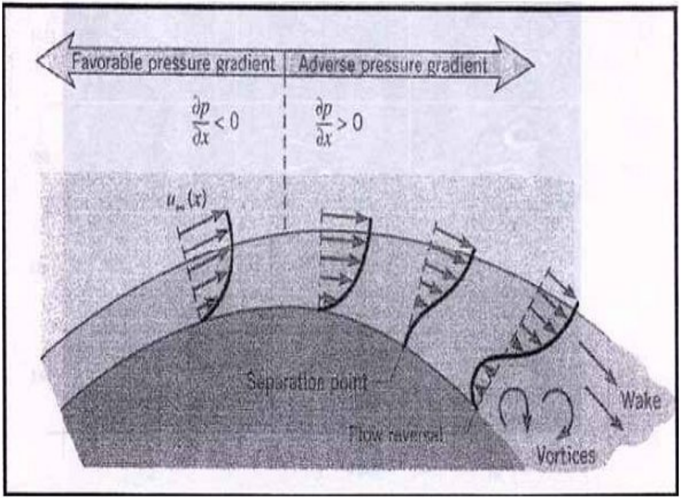


Figure 1. Décollement de la couche limite

Cauchy problem for Prandtl

Positive results:

- Analytic data (w.r.t. x): *locally well-posed* ([Sammartino and Caflisch 1998], [Lombardo-Cannone and Sammartino 2003], [Kukavica and Vicol 2012]).
- Monotonic data (w.r.t. y): *locally well-posed, globally if* $\partial_x P_E < 0$ ([Oleinik 1968], [Xin-Zhang 2004] (Crocco transform). [Alexandre-Wang-Xu and Yang 2012], [Masmoudi and Wong 2012]).

Questions: Non-monotonic data ? Sobolev theory ?

Studied in [Gérard-Varet and Dormy 2010]. For better understanding:

Linearization around a shear flow $(U(y), 0)$.

$$\begin{cases} \partial_t u + U\partial_x u + vU' - \partial_y^2 u = 0, & \text{in } \mathbb{T}\times\mathbb{R}^+. \\ \partial_x u + \partial_y v = 0, & \text{in } \mathbb{T}\times\mathbb{R}^+, \\ (u,v)|_{y=0} = (0,0), & \lim_{y\to+\infty} u = 0. \end{cases} \qquad \text{(PL)}$$

Theorem (Gevrey well-posedness [Gérard-Varet and M])

Let $\tau_0 > 0$, $s \gg 1$ *even*, $\sigma \geq \gamma + \frac{1}{2} \gg 1$. *Let*

$$u_0 \in G^{7/4}_{\tau_0}(\mathbb{T};\, H^{s+1}_{\gamma-1}), \quad \omega_0 := \partial_y u_0 \in G^{7/4}_{\tau_0}(\mathbb{T};\, H^s_\gamma),$$

satisfying: $u_0|_{y=0} = 0$, *as well as* (H1), (H2). *Then there exists* $T > 0$, $0 < \tau \leq \tau_0$ *and a unique solution*

$$u \in L^\infty\big(0, T; G^{7/4}_\tau(\mathbb{T};\, H^{s+1}_{\gamma-1})\big), \quad \omega \in L^\infty\big(0, T; G^{7/4}_\tau(\mathbb{T};\, H^s_\gamma)\big),$$

of (P), *with initial data* u_0.

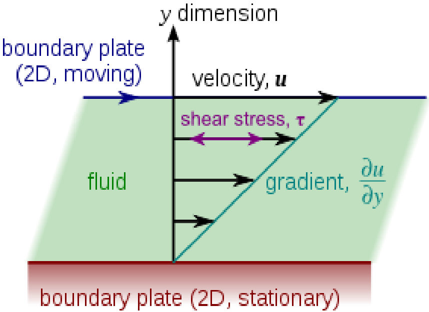
H^s_γ is a weighted Sobolev space.

Two main points :

- The Gevrey requirement is due to an instability in the linearized problem
- The proof used the physical space representation of the Gevrey norm because the cancellation of Tak Kwong Wong and M is given in physical space.

Sommerfeld paradox and Orr mechanism

Sommerfeld (1868-1951) paradox (or turbulence paradox) says that mathematically the Couette flow (linear shear) is linearly stable (spectral stability) for all Reynolds numbers, but experimentally transition to turbulence is observed under perturbations of any size when the Reynolds number is large.



One of the main explanation was given by W. Orr (1866-1934) in 1907 and is based on the Orr mechanism.

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- Experiments with fluids have been unable to sustain planar Couette flow at high Reynolds numbers despite the spectral stability.
- We will be studying this question in the simplest setting: $(x, y) \in \mathbb{T} \times \mathbb{R}$.

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- The decay of the velocity field in 2D Euler is referred to as *inviscid damping*.

Stability and Instability in hydrodynamics

Many mathematical results

- C.C.-Lin
- Drazin-Howard and Drazin-Reid
- Arnold
- Friedlander-Strauss-Vishik
- Grenier
- Bardos-Guo-Strauss
- Z. Lin

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- The vorticity ω, and the velocity $\nabla^\perp\psi$ it creates through the Biot-Savart law, are the perturbation from the background shear flow.

Mixing as a linear decay estimate

- Linearizing 2D Euler around Couette flow gives the passive transport:

$$\omega_t + y\partial_x\omega = 0. \tag{4}$$

- Hence $\omega(t, x, y) = \omega_0(x - ty, y)$.

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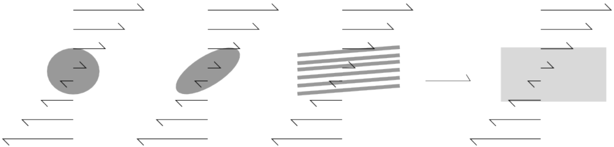

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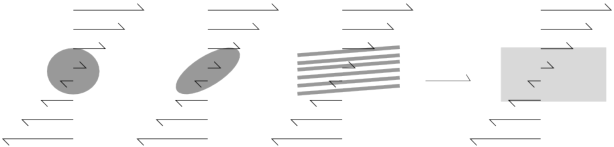

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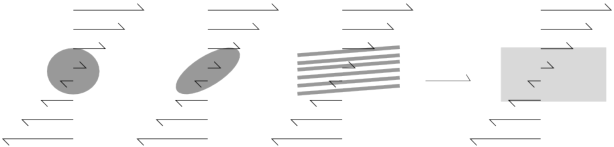

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  - (a) $\omega(t) \rightharpoonup \int \omega_0(x,y)dx$ (the trajectories are not pre-compact - lose information to small scales as $t \to \infty$)
  - (b) $\partial_y \omega(t,x,y) = O(t)$ but $\partial_x \omega$ and $(\partial_y + t\partial_x)\omega$ are bounded.

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- The transient amplification that occurs at $\eta = kt$ and subsequent decay is called the *Orr mechanism*. We call the time $t = \frac{\eta}{k}$ the 'Orr critical time'.

Recall that in the moving frame the streamfunction is given by

$$\hat{\phi}(k,\eta) = -\frac{\hat{\omega}_0(k,\eta)}{k^2 + |\eta - kt|^2}.$$

Consider a pure plane wave with $\eta \gg k$ being sheared (based on a picture of Boyd):

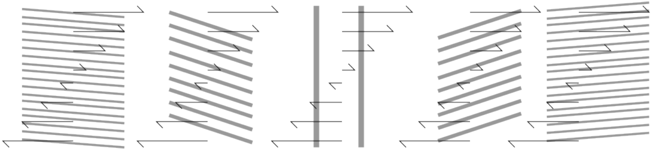

Figure: The center image occurs at the *critical time* $t = \eta/k$

.

The right half is mixing and losing kinetic energy but the left-half is *un-mixing*.

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- Lin and Zeng show in 2011 that small perturbations to the vorticity in H^s do not necessarily return to any shear flow if $s < 3/2$ (they can be trapped in Kelvin's cat's eye vortices).

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- Hence our initial data is Gevrey-$\frac{1}{s}$.
- This regularity class is a bit obscure, but the proof will reveal why it is natural for this problem.
- We also have to take mean-zero and well-localized: $\int \omega_0 dxdy = 0$ and $\int y^2 |\omega_0(x,y)| \, dxdy$ sufficiently small.

Asymptotic stability of nearly-Couette shear flows

Theorem (Bedrossian - Masmoudi 2013)

For all $1/2 < s \leq 1$, $\lambda > \lambda' > 0$, *there exists an* $\epsilon_0 = \epsilon_0(\lambda, \lambda', s) \leq 1/2$ *such that if*

$$\|\omega_0\|_\lambda = \epsilon < \epsilon_0,$$

then the vorticity mixes like passive advection in a shear flow up to a logarithmic phase correction as $t \to \infty$, *in the sense that: there exists an* f_∞, $u_\infty(y)$ *and* $\theta(t, y) = O(\log t)$ *such that*

$$\|\omega(t, x + ty + u_\infty(y)t + \theta(t, y), y) - f_\infty(x, y)\|_{\lambda'} \lesssim \frac{1}{t}. \tag{5}$$

Moreover, the velocity field U *converges strongly in* L^2 *to the shear flow* $(y + u_\infty(y), 0)$:

$$\|U^x(t) - u_\infty\|_2 \lesssim \frac{1}{\langle t\rangle} \tag{6a}$$

$$\|U^y(t)\|_2 \lesssim \frac{1}{\langle t\rangle^2}, \tag{6b}$$

2D Euler is quasi-linear

- The evolution of the vorticity is asymptotically like passive transport in a shear flow:

$$\omega(t,x,y) \sim f_\infty\left(x - ty - u_\infty(y)t - \theta(t,y), y\right), \quad \text{when } t \to \infty.$$

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- In this context, the Euler equations should be considered "*quasi-linear*" (whereas the Vlasov equations are "*semi-linear*").
- This is especially dangerous since we need regularity to get the damping...but we will only have bounded derivatives in certain directions *determined by the background shear flow that we don't know*.

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- The change $y \to v$ is to ensure that the Biot-Savart law in the new variables has the same Orr critical times.

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- Define $f(t,z,v) = \omega(t,x,y)$ and the transformed streamfunction $\phi(t,z,v) = \psi(t,x,y)$:

$$\partial_t f + \partial_t v \partial_v f + \partial_y v \nabla^{\perp}_{z,v}(\phi - <\phi>) \cdot \nabla_{z,v} f = 0 \tag{8a}$$

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- Upon changing variables back, this will imply the kind of damping we want...

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- Since Orr's work, the unresolved fundamental question about the Couette flow is whether the Orr mechanism always drives instability or whether or not stability can still hold under some hypotheses (Orr pondered on this question too).
- Clearly, we need to have a good understanding of how the Orr mechanism manifests in the nonlinear problem.

Paraproducts as ‘linearization’

- Divide the nonlinearity based on the relative frequencies of the two factors (known as a *paraproduct* - introduced by Bony),

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- We can estimate the contribution from transport by adapting the Gevrey regularity methods of Foias/Temam, Levermore/Oliver/Titi, Kukavica/Vicol...
- Since the velocity field is in 'low frequency' we can (mostly) avoid dealing with the Orr critical times.

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- Echoes are actually a special case of a potentially much worse repeating cascade of information to modes which are *unmixing* (a scenario studied by Vanneste et. al., Trefethen et. al., Waleffe, Baggett et. al. etc mostly in 3D).

Nonlinear interactions and the Orr Mechanism

- With the paraproduct in mind, we should use a model problem in which f evolves linearly, interacting with a background f_{lo}, something like:

$$\partial_t \hat{f}(t,k,\eta) = \sum_{l \neq 0} \int_\xi \xi \hat{\phi}(t,l,\xi)(k-l)\hat{f}_{lo}(t,k-l,\eta-\xi)d\xi.$$

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- $(\nabla f)_{lo}$ reduces interactions of well-separated frequencies in v, so let's think of η as a parameter:

$$\partial_t f(t,k,\eta) = \sum_{l \neq 0} \frac{\eta \hat{f}(t,l,\eta)}{l^2 + |\eta - lt|^2}(k-l)\hat{f}_{lo}(t,k-l,0).$$

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$$\partial_t f_C \approx \frac{k^2}{|\eta|} f_{NC} \tag{12a}$$

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- This suggests that as time goes to infinity, the solution (in these new variables) could lose a large amount of Gevrey-2 regularity (hence the requirement $s > 1/2$).

Energy Estimates

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  - we do not completely exhaust all of our regularity as $t \to \infty$.

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- Instead, we build the behavior of the toy model *into the energy estimate.*
- The key idea is to design a norm which gets weaker *in the right modes at the right times* so that
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  - Transport vs reaction.
  - Need to also get estimates on the time evolution of the coordinate system...

$$\|A(t,\nabla)f\|_2^2 = \sum_k \int_\eta |A(t,k,\eta)f(t,k,\eta)|^2 \, d\eta.$$

The multiplier A has several components :

$$A_k(t,\eta) = e^{\lambda(t)|k,\eta|^s} \langle k,\eta\rangle^\sigma J_k(t,\eta) B_k(t,\eta).$$

The index $\lambda(t)$ is the bulk Gevrey$-\frac{1}{s}$ regularity and wil be chosen to satisfy

$$\dot{\lambda}(t) = -K_\lambda \frac{\epsilon}{\langle t\rangle^q}(1+\lambda(t)),$$

for some K_λ and $q > 1$.
The main multiplier for dealing with the Orr mechanism and the nonlinear growth it yields is

$$J_k(t,\eta) = \frac{e^{\mu|\eta|^{1/2}}}{w_k(t,\eta)} + e^{\mu|k|^{1/2}}$$

where $w_k(t,\eta)$ describes the expected 'worst-case' growth due to nonlinear interactions at the critical times

With this special norm, we can define our main energy:

$$E(t) = \frac{1}{2}\|Af\|_2^2 + \langle t\rangle^{4-\epsilon}\|\frac{A}{\langle\partial_v\rangle^s}[\partial_t v]\|_2^2. \tag{13}$$

In a sense, there are two coupled energy estimates we need to make: the one on Af and the one on $A\partial_t v$.

$$\frac{1}{2}\frac{d}{dt}\int |Af|^2\, dx = -CK_\lambda - CK_w - CK_B + \int AfA(u\cdot\nabla f)dx, \qquad (14)$$

where the CK stands for 'Cauchy-Kovalevskaya' since these three terms arise from the progressive weakening of the norm in time, and are expressed as

$$CK_\lambda = -\dot{\lambda}(t)\| \,|\nabla|^{s/2} Af\|_2^2$$
$$CK_w = \sum_k \int \frac{\partial_t w_k(t,\eta)}{w_k(t,\eta)} \left|A_k(t,\eta)\hat{f}_k(t,\eta)\right|^2 d\eta$$
$$CK_B = -\sum_k \int \frac{\partial_t B(t,\eta)}{B(t,\eta)} \left|A_k(t,\eta)\hat{f}_k(t,\eta)\right|^2 d\eta.$$

The rest of the proof is to control Transport and Reaction terms....

We still need to prove :

- Elliptic estimates to invert $\Delta_t \phi = f$
- Control of $[\partial_t v]$ which appears in $E(t)$

Mixing in the Vlasov equations

- The *collisionless* Vlasov equations are the kinetic model for a probability distribution $f(t,x,v) : \mathbb{T}^d \times \mathbb{R}^d \to [0,\infty)$:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + F(t,x) \cdot \nabla_v f = 0 \\ F(t,x) = -\nabla_x W *_x \left(\int f(t,\cdot,v) dv - 1 \right), \end{cases} \tag{15}$$

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- Predicts asymptotic stability (in some sense) without dissipation, entropy production etc of any kind.

Mixing by kinetic free transport

- The fundamental mechanism is the same as in Euler: the *phase mixing* (mixing in phase-space) due to the free transport $\partial_t f + v \cdot \nabla_x f = 0$ (van Kampen '55).

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- Landau damping and asymptotic stability for the nonlinear Vlasov equations was proved by Mouhot and Villani in 2011 - the first result of its kind.
- We have a new proof of this result (slightly more general) with J. Bedrossian and C. Mouhot.

Landau damping in the Vlasov equations

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- $f^0(v)$ is an equillibrium so we can study mean-zero perturbations $f(t,x,v) = f^0(v) + h(t,x,v)$

$$\begin{cases} \partial_t h + v \cdot \nabla_x h + F(t,x) \cdot \nabla_v h + F(t,x) \cdot \nabla_v f^0 = 0, \\ F(t,x) = -\nabla_x W *_x \int h(t,\cdot,v) dv, \\ h(t=0,x,v) = h_{in}(x,v). \end{cases} \tag{16}$$

Nonlinear Landau damping

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Theorem (Bedrossian, Masmoudi, Mouhot 2013)

Let f^0 satisfy a suitable linear stability condition (but not necessarily 'small'), $(2+\gamma)^{-1} < s \leq 1$, $M > d$ be an integer, and $\lambda_0 > \lambda' > 0$ be arbitrary. Then there exists an $\epsilon_0 = \epsilon_0(d, M, f^0, \lambda_0, \lambda', s)$ such that if h_{in} is mean zero and

$$\sum_{\alpha \in N^d : |\alpha| \leq M} \| v^\alpha h_{in} \|^2_{\lambda_0;s} < \epsilon^2 \leq \epsilon_0^2,$$

then there exists a mean-zero f_∞ satisfying

$$\| h(t, x+vt, v) - f_\infty(x,v) \|_{\lambda';s} \lesssim \epsilon e^{-\frac{1}{2}(\lambda_0 - \lambda')t^s}, \tag{17a}$$

$$\| e^{\lambda' \langle k, kt \rangle^s} \hat{\rho}_k(t) \|_{L^2_k} \lesssim \epsilon e^{-\frac{1}{2}(\lambda_0 - \lambda')t^s}. \tag{17b}$$

Ideas of proof : We introduce a multiplier A

$$A_k(t,\eta) = e^{\lambda(t)\langle k,\eta\rangle^s}\langle k,\eta\rangle^{\sigma},$$

where $\sigma > d+8$ is fixed and $\lambda(t)$ is an index (or 'radius') of regularity which is decreasing in time.
Landau damping predicts that the solution evolves by kinetic free transport as $t \to \infty$:

$$h(t,x,v) \sim f_\infty(x - vt, v).$$

We 'mod out' by the lack of compactness of the free transport and work in the coordinates $z = x - vt$ with $g(t,z,v) = h(t,x,v)$. Then (17a) becomes equivalent to $g(t) \to f_\infty$ strongly in Gevrey$-\frac{1}{s}$.

We use a Bootstrap argument to propagate the following control:

$$\sum_{\alpha\in\mathbb{N}^d:|\alpha|\leq M} \|\langle\nabla_{z,v}\rangle A(v^\alpha g)(t)\|_2^2 \leq 4K_1\langle t\rangle^7\epsilon^2 \tag{18a}$$

$$\sum_{\alpha\in\mathbb{N}^d:|\alpha|\leq M} \|\langle\nabla_{z,v}\rangle^{-\beta} A(v^\alpha g)(t)\|_2^2 \leq 4K_2\epsilon^2 \tag{18b}$$

$$\int_0^t \|A\rho(\tau)\|_2^2 d\tau \leq 4K_3\epsilon^2, \tag{18c}$$

For the first two estimates, we use an energy method (like in Euler)
For the third, we use the stability assumption of the background.

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- Damping asymmetries in radial vortices?
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- Do any of these ideas apply to Vlasov ? (with Bedrossian and Mouhot)

Thank you for your attention!