Relative entropy and contraction to extremal shocks for conservation laws

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The equation

- Full compressible Euler system (polytropic gas):

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P) = 0 \\ \partial_t(\rho E) + \partial_x(\rho u E + uP) = 0, \end{cases}$$

where $E = \frac{1}{2}u^2 + e$. The equation of state for a polytropic gas is given by $(\gamma > 1)$

$$P = (\gamma - 1)\rho e$$

- Isentropic gas dynamics $(\gamma > 1)$:

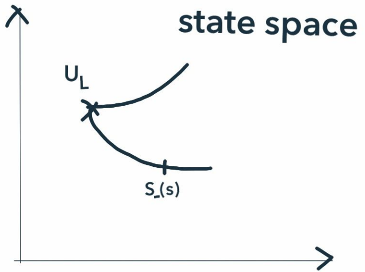
$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0,$$
$$\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla \rho^\gamma = 0.$$

Main goal

- We consider shocks, that it discontinuous, piecewise constant solutions.
- We restrict ourselves to the 1D case.
- We are interested in contraction properties of those special solutions.
- It is closely related to the study of asymptotic limits to shocks (for instance, from Navier-Stokes to Euler).
- Remark: We can consider more general systems than the Euler case.

Extremal shocks

For general system: $\partial_t U + \partial_x f(U) = 0$.
Curves of U_R such that (U_L, U_R, σ) is a shock.



for $s \approx 0$, $\sigma \approx \lambda_i(U_L)$ where $\lambda_i(U_L)$ is the i th eigenvalue of $f'(U)$.

A physical motivation

- Shocks are fundamental solutions in physics. But, the derivation of the macroscopic model is problematic for those solutions (no local thermodynamical equilibrium for the derivation from kinetic equations, for instance).

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- Shocks are fundamental solutions in physics. But, the derivation of the macroscopic model is problematic for those solutions (no local thermodynamical equilibrium for the derivation from kinetic equations, for instance).
- The difficulty come from the production of layers.
- What happens if the system carries too much energy for the stability of the layer ?

Mathematical motivations

- In 1D, Shocks are the scale invariant solutions of the equation.

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- In PDE, the stability of scale invariant solutions is fundamental for the study of the behavior of general solutions.
  - Parabolic equations (regularity): Kohn and nirenberg, Caffarelli....
  - dispersive equations (blow-ups): Merle, Koenig...
  - Conservations laws: strong traces, well-posedness of solutions and asymptotic limits: Bressan, Liu...
- Remark: For conservation laws, it is based on L^1 stability of the shocks.

Entropy

The system in play have entropies which are strictly convex with respect to the conserved quantities.

- Full Euler system:

$$\eta(\rho, \rho u, \rho E) = (\gamma - 1)\rho \ln \rho - \rho \ln e$$

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- The Isentropic Euler system has also a convex entropy (which is the physical energy):

$$U = (\rho, \rho u), \qquad \eta(U) = \rho u^2/2 + \rho^\gamma/(\gamma - 1).$$

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$$U = (\rho, \rho u), \qquad \eta(U) = \rho u^2/2 + \rho^\gamma/(\gamma - 1).$$

- To be an entropy means that any physical solutions verify:

$$\int \eta(U(t,x))\, dx$$

is not increasing.

Relative entropy

We define the relative entropy between two states $U_1, U_2 \in \mathcal{V}$

$$\eta(U_1|U_2) = \eta(U_1) - \eta(U_2) - \eta'(U_2)(U_1 - U_2).$$

If η is strictly convex then

$$\eta(U_1|U_2) \approx |U_1 - U_2|^2.$$

Dafermos- DiPerna (79'): If U_2 is a Lipschitz solution and U_1 is a weak solution, then

$$\frac{d}{dt}\int_{\mathbb{R}} \eta(U_1|U_2)\,dx \leq C(U_2)\int_{\mathbb{R}} \eta(U_1|U_2)\,dx.$$

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Especially, if at $t = 0$ $\int_{\mathbb{R}} \eta(U_1|U_2)\,dx \approx \varepsilon^2$, then at t: $\approx e^{Ct}\varepsilon^2$.

Strong stability L^2 (I)

- It implies a STRONG stability of Lipschitz solutions in L^2.
- Weak/strong uniqueness, Dafermos DiPerna, Lions, Brenier, Feireisl....
- Can be used for asymptotic limit and hydrodynamic. In other context, see Yau (91'), Bardos Golse Levermore (91'),...

Strong stability L^2 (II)

- In this context, the consistence implies the convergence. The nonlinearities are driven by the strong stability of the limit function.

Strong stability L^2 (II)

- In this context, the consistence implies the convergence. The nonlinearities are driven by the strong stability of the limit function.
- Can we follow the same strategy for shocks ?

The scalar case

For SCALAR conservation laws, Kruzkov theory provides a L^1 contraction: For any weak entropic solutions u, v:

$$\|u(t) - v(t)\|_{L^1(\mathbb{R})} \leq \|u(0) - v(0)\|_{L^1(\mathbb{R})}.$$

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But it is not true for L^2 !!

Problem with shocks and L^2 theory

- The contraction in L^2 is NOT valid for shocks.

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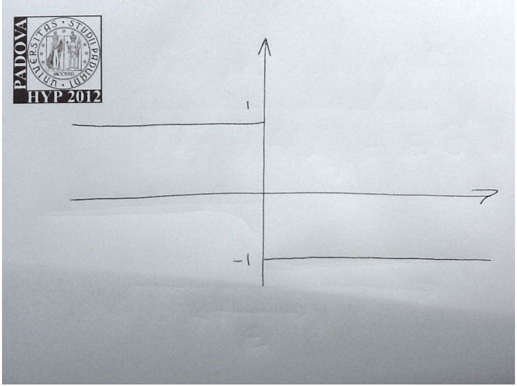
- Example for Burgers:

$$\partial_t u + \partial_x u^2 = 0.$$



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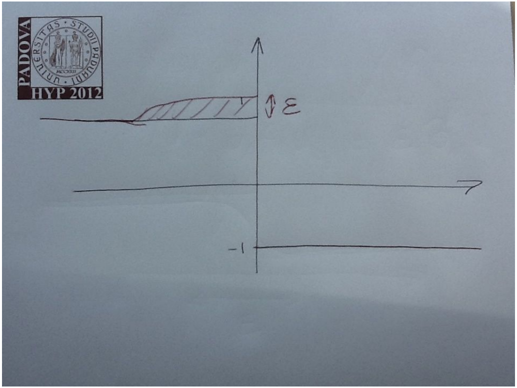

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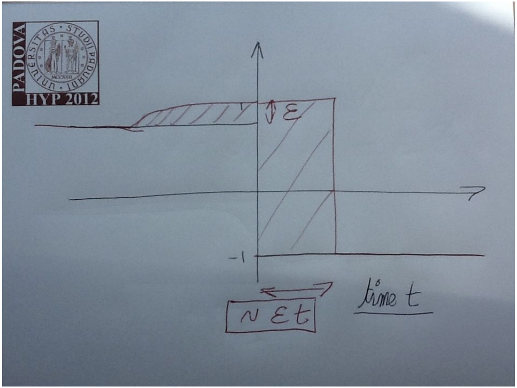

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- An ε perturbation of a shock S at $t = 0$ will give an error $\approx \sqrt{\varepsilon t}$ at time t.
- This is because it perturbs the SPEED of the shock.
- However, the perturbation of the profile of the shock decreases (up to a translation).

Contraction up to a shift for the scalar case

Theorem

(Leger) Consider (U_L, U_R, σ) a shock. Then there exist a constant $C > 0$ such that for any $U^0 \in L^2(\mathbb{R})$ there exists a Lipschitzian map $x(t)$ such that:

$$\int_{-\infty}^{\infty} |U(t,x) - S(x - x(t))|^2 \, dx \le \int_{-\infty}^{\infty} |U^0(x) - S(x)|^2 \, dx$$

$$|x(t) - \sigma t| \le C\sqrt{t(1+t)} \|U^0 - S\|_{L^2}.$$

For $x < 0$, $S(x) = U_L$, for $x > 0$, $S(x) = U_R$.

General systems

We consider a strictly hyperbolic system of conservation laws

$$\partial_t u + \partial_x f(u) = 0, \qquad u(t,x) \in \mathbb{R}^n. \tag{1}$$

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We want to investigate whether we can expect some contraction to shocks with respect to the relative entropy, up to a shift.
Consider $S(x) = u_L$ for $x < 0$ and $S(x) = u_R$ for $x > 0$, where (u_L, u_R, σ) is a shock.
Do we have for any weak solution u of (1), the existence of $t \to X(t)$ such that

$$\int_{\mathbb{R}} \eta(u(t,x)|S(x - X(t)))\, dx \leq \int_{\mathbb{R}} \eta(u_0(x)|S(x))\, dx?$$

Criteria for systems (Serre, V.) (I)

Let (η, q) be a entropy/entropy flux for the system.
Let (u_L, u_R, σ) a given shock. For any function g, we denote $[g] = g(u_R) - g(u_L)$.

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- For any fixed state $u \in \mathbb{R}^n$, we denote

$$D_{sm} = [\eta' \cdot f - q] - [\eta'] \cdot f(u).$$

- For any other entropic discontinuity (u_-, u_+, h), we denote

$$D_{RH} = [\eta' \cdot f - q] - h[\eta' \cdot u - \eta] + q_+ - q_- - h(\eta_+ - \eta_-) - [\eta'] \cdot (f - hu)_\pm.$$

Criteria for systems (Serre, V.) (II)

- Let Π be the hyperplane $\{u \in \mathbb{R}^n | \eta(u|u_L) = \eta(u|u_R)\}$.
- We say that (u_L, u_R, σ) is relative-entropy stable, if $D_{sm} \leq 0$ on Π, and if for every (u_-, u_+, h) entropy discontinuity such that u_- and u_+ are separated by Π, we have $D_{RH} \leq 0$.

Theorem

(Serre, V.) If (u_L, u_R, σ) is relative-entropy stable, if and only if, for any weak solution u, there exists a shift $x(t)$ such that for any $t > 0$

$$\int_{\mathbb{R}} \eta(u(t)|S(\cdot - x(t))\, dx \leq \int_{\mathbb{R}} \eta(u_0|S)\, dx.$$

Euristic of the proof (I)

- We conpute at each time the minimum of

$$h \mapsto E(u(t); h) := \int_{-\infty}^{h} \eta(u|u_L)dx + \int_{h}^{\infty} \eta(u|u_R)dx$$

and consider the evolution of this minimum as time increases.

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  and consider the evolution of this minimum as time increases.
- the minimum is therefore achieved at some finite $h = h(t)$, where we have

$$\left.\frac{d}{dh}\right|_{h(t)-0} E \leq 0 \leq \left.\frac{d}{dh}\right|_{h(t)+0} E$$

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- If u is continuous at $h(t)$ then $u \in \Pi$. Else, u_- is on the same side of Π as u_L, and u_+ is on the same side as u_R.

Euristic of the proof (I)

If u is continuous at $h(t)$, $\eta(u|u_L) - \eta(u|u_R) = 0$ at $h(t)$.

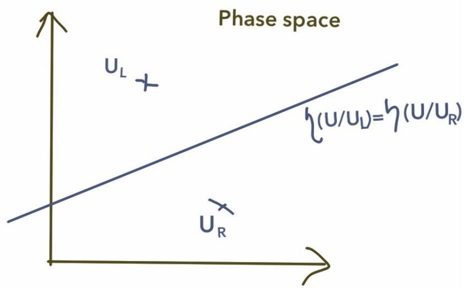

Figure: Continuous case

Euristic of the proof (I)

If u is not continuous at $h(t)$, (u_-, u_+) is a shock such that u_- is on the same side of Π as u_L, and u_+ is on the same side as u_R.

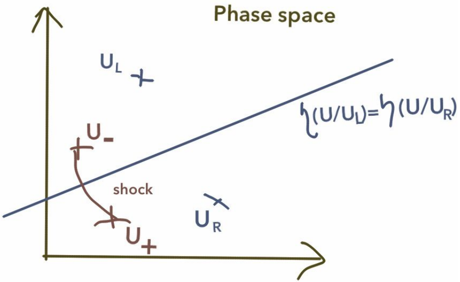

Figure: Discontinuous case

Euristic of the proof (II)

- However $t \longrightarrow h(t)$ may be not continuous.
- instead, we solve an ODE for $t \longrightarrow x(t)$ in the sense of Filippov...

Example of systems (Serre, V.)

- The Keyfitz-Kranzer systems are relative-entropy stable for ϕ decreasing and $\rho \longrightarrow \rho\phi(\rho)$ convex.

$$\partial_t u + \partial_x(\phi(|u|)u) = 0, \qquad u(t,x) \in \mathbb{R}^n.$$

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- The Euler systems are NOT !!!
- But we have still a contraction if we change the pseudo-norm.

Contraction for shocks

Theorem

(V.) Consider (U_L, U_R, σ) *a extreme shock. Then there exists a constant* $a > 0$ *depending only on the shock with the following property.*

Consider any $K > 0$*. There exists* $C_K > 0$ *such that, for any weak entropic solution such that* $\|U\|_{L^\infty} \leq K$*, there exists a Lipschtiz path* $x(t)$ *such that for any* $t > 0$*, the pseudo norm*

$$\int_{-\infty}^{0} \eta(U(t, x + x(t))|U_L)\, dx + a \int_{0}^{\infty} \eta(U(t, x + x(t))|U_R)\, dx$$

is non increasing in time. Moreover, for every $t > 0$*:*

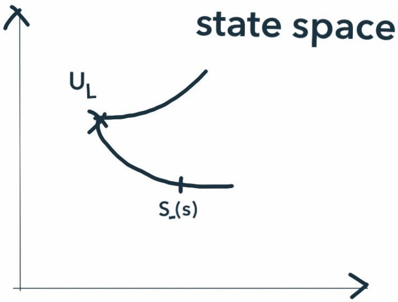
$$|x'(t)| \leq C_K, \qquad |x(t) - \sigma t| \leq C_K \sqrt{t} \|U_0 - S\|_{L^2(\mathbb{R})}.$$

Remarks

- Provides a contraction result in the class of bounded weak solutions having a strong trace property. There is no smallness conditions. We do not need the microstructure of the solutions. The contraction is driven by the entropy.
- The pseudo-norm depends only on the system and on the fixed shock (U_L, U_R, σ). It does not depend on the weak solution U. Only the drift $x(t)$ depends on the L^∞ norm of U.

Hypotheses

- Shocks is extremal and verifies the Liu property: $\frac{d}{ds}\sigma(s) < 0$
- $\frac{d}{ds}\eta(U_L|S_-(s)) > 0.$



Hypotheses

Surprisingly, if the increase of shock along the shock curve is measured with the entropy (instead of the relative entropy), shocks can be even not stable (see Freistuhler-Zumbrun). However, when measured by the relative entropy, it verifies the Lopatinski conditions (Texier-Zumbrun).

the drift (1)

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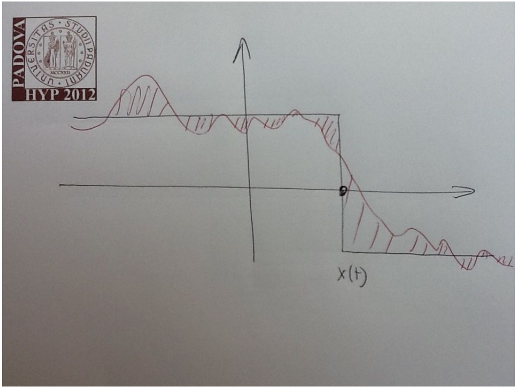

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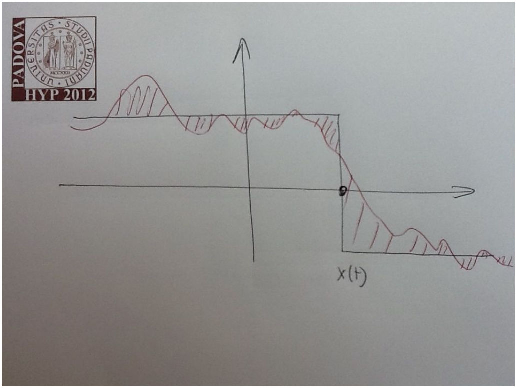

By choosing $x'(t)$ we can change the fluxes of entropy (depending on the "value" of $U(t, x(t))$!).

the drift (2)

- We will solve an ODE with a discontinuous flux. We use Fillipov flow.
- Generically, the interface $x(t)$ is stuck in a shock !

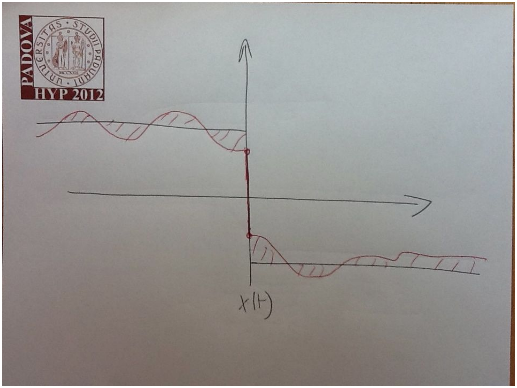

Figure: Drift

drift(3)

- We can construct $x(t)$ such that the only relevant shocks (U_-, U_+) stuck at the interface are of the same family than (U_L, U_R).
- Moreover $\eta(U_-|U_L) \leq a\eta(U_-|U_R)$ and $\eta(U_+|U_L) \leq a\eta(U_+|U_R)$.

Non-homogeneous pseudo-distance

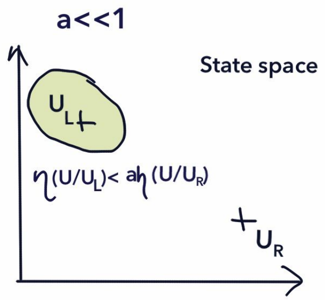

Figure: Drift

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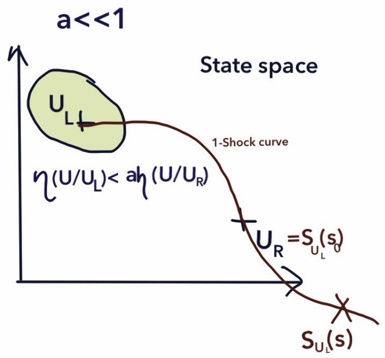

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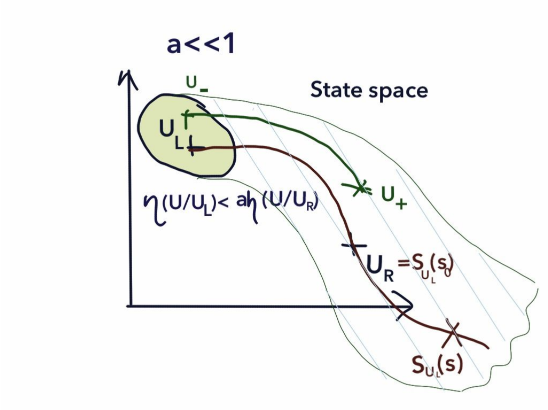

Figure: Drift

Shocks and shock layers

- Asymptotic limits to shocks involve the production of LAYERS.

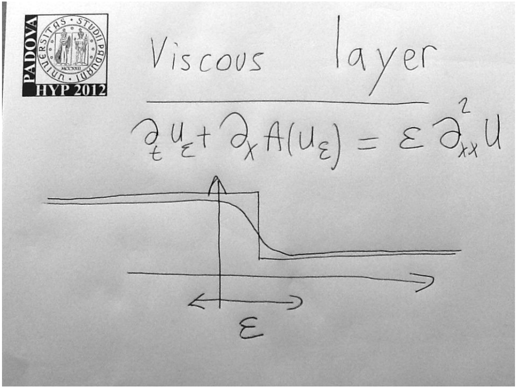

Figure: example of layer

Shocks and shock layers

- Asymptotic limits to shocks involve the production of LAYERS.
- The control of the layers usually involves smallness conditions: Liu Zumbrun, Bressan (L^1 theory)...

Shocks and shock layers (2)

QUESTION:

- Is the whole structure of the layer needed to perform asymptotic limits ?
- Would the entropy (relative entropy) be enough to drive the convergence, whatever the fine structure in the layer ?
- Do we have enough strong stability on shocks ?

A first application to asymptotic analysis

Scalar case:

$$\partial_t U_\varepsilon + \partial_x U_\varepsilon^2 = \varepsilon \partial_{xx} U_\varepsilon.$$

For U_L, U_R, we define $S(x) = U_L$ if $x < 0$, and $S(x) = U_R$ if $x > 0$.

The result

Theorem

(Choi, V.) There exists $\varepsilon_0 > 0$*, such that for any* U_ε *solution to the viscous Burgers equation with* $\varepsilon < \varepsilon_0$ *and*

$$\|(\partial_x U_0)_+\|_{L^2} \leq C,$$

there exists $X(t)$ *Lipschitz such that for any time* $t > 0$

$$\int \eta(U_\varepsilon(t,x)|S_0(x - X(t)))\,dx$$
$$\leq \int \eta(U_0(x)|S_0(x))\,dx + C\varepsilon(\log^+(1/\varepsilon) + 1)(1 + t).$$

Case with small initial perturbation

- If $\int \eta(U_0(x)|S(x))\,dx \leq C\varepsilon$, Then we can study the layer problem by scaling $V(t,x) = U(\varepsilon t, \varepsilon x)$.

$$\partial_t V + \partial_x V^2 = \partial_{xx} V.$$

- This problem has been extendedly studied (Ilin Oleinik (64'), Osher and Ralston (82'), Goodman (89'), Jones Gardner and Kapitula (93'), Freistuhler and Serre (96'), Kenig and Merle (06'))
- V converges to the layer $Q(x - \sigma t)$ up to a drift (nondependent on time).

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- V converges to the layer $Q(x - \sigma t)$ up to a drift (nondependent on time).
- In this context, our result is weaker (the error is bigger than ε).
- But, in the case $\int \eta(U_0(x)|S(x))\,dx >> \varepsilon$, the layer study collapse. (The layer can be destroyed). Still, we can obtain the expected limit with a precise rate.

Citations

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- V. (13): system case without the restriction.
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Thank you

THANK YOU !