Easy-to-State but Hard to Solve: Challenges in Polyhedral Computation

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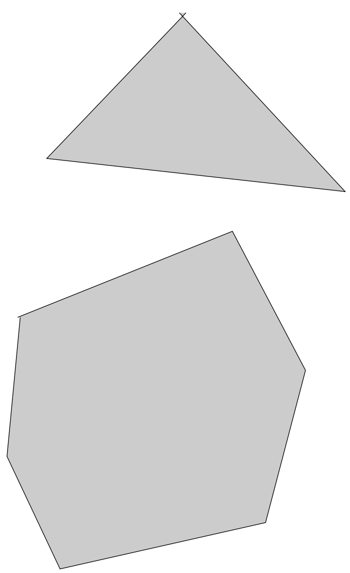
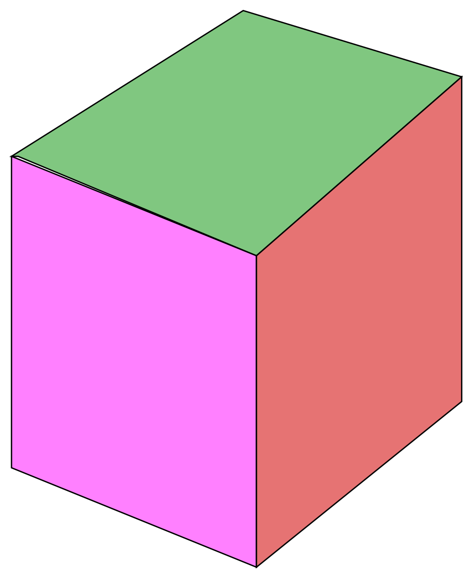
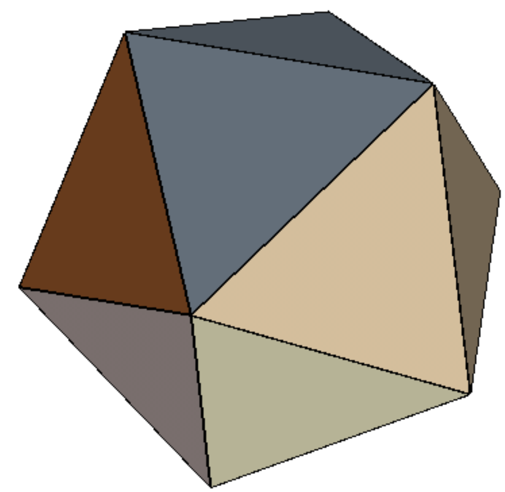
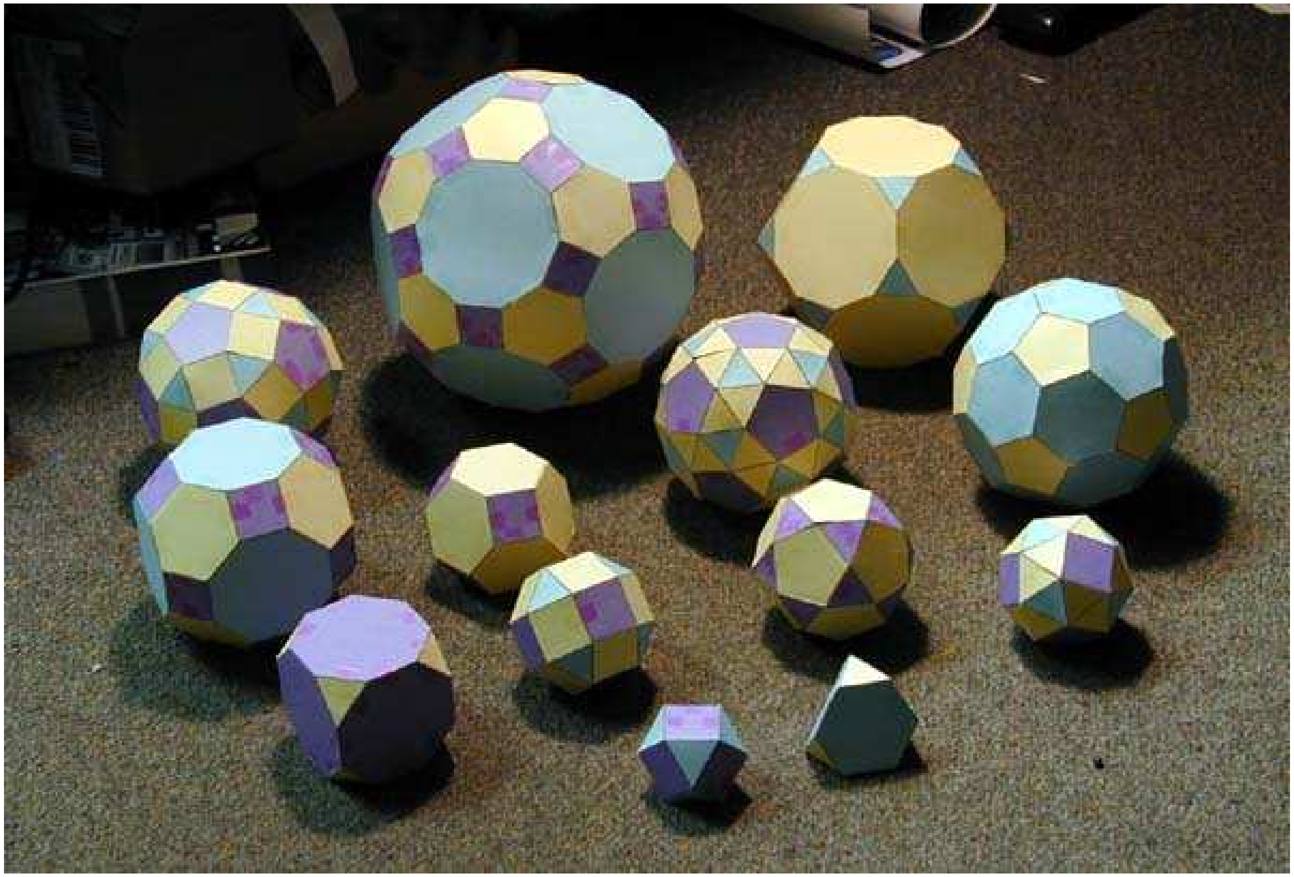




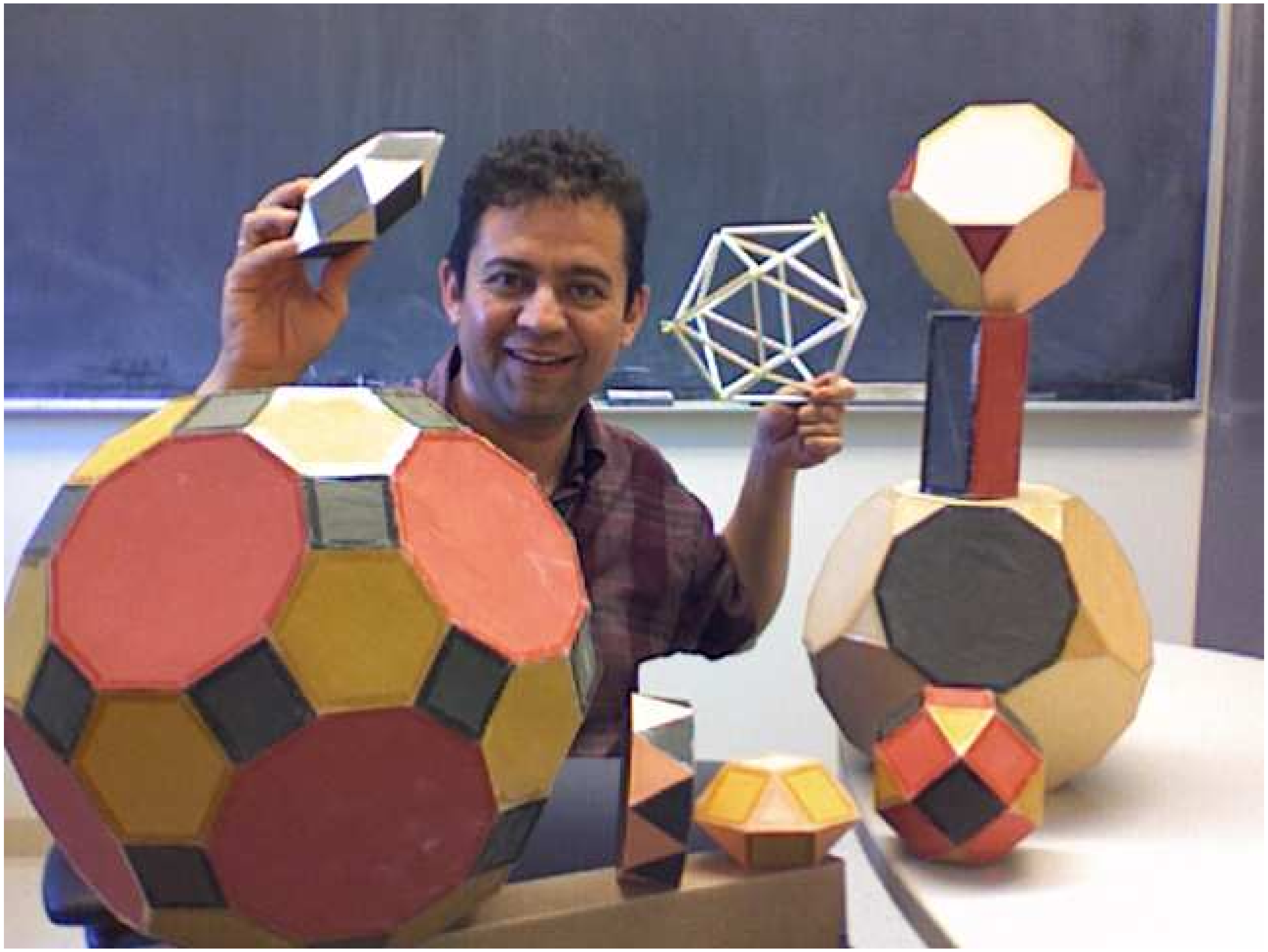


What is a Convex Polytope?

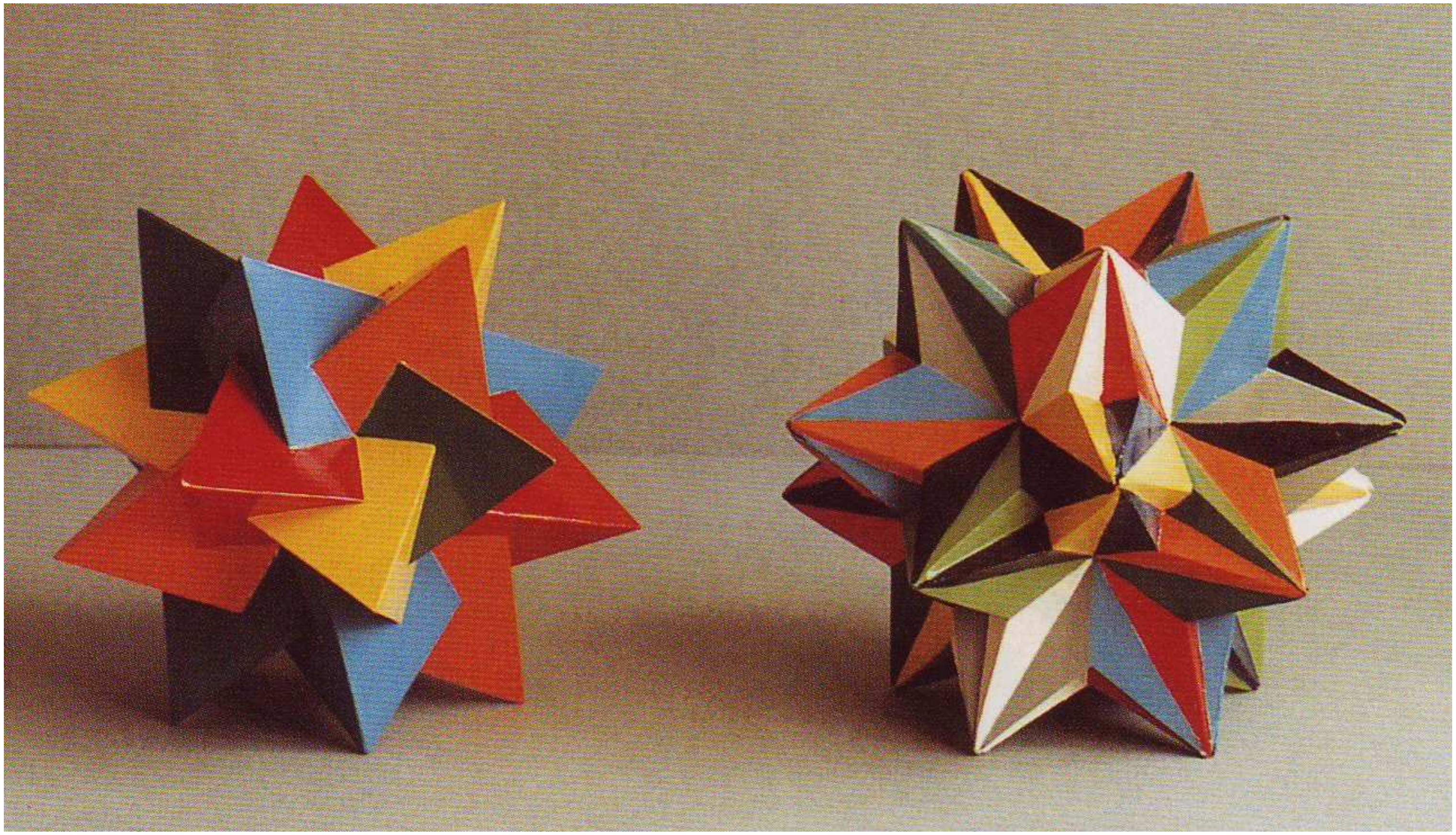
Well,something like these...



or like these

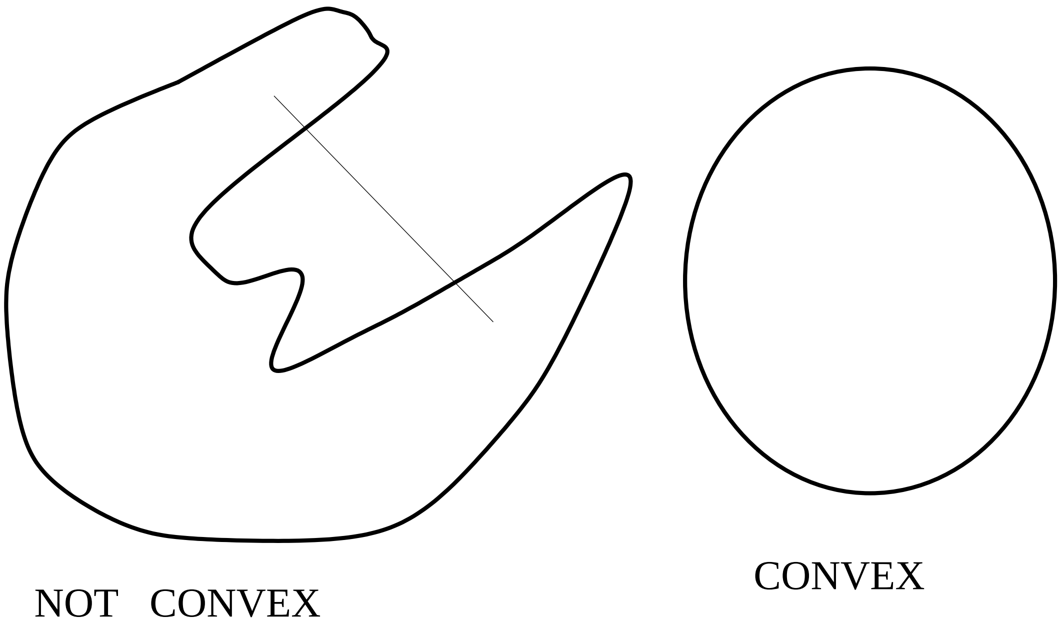


But NOT quite like these!

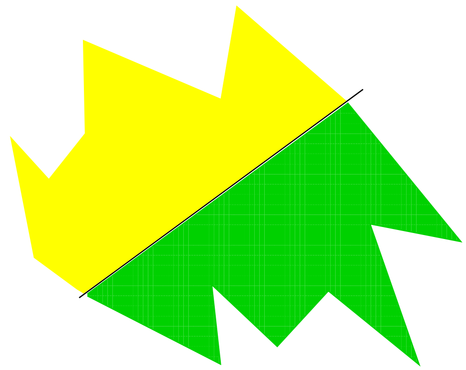


A definition PLEASE!

The word CONVEX stands for sets that contain any line segment joining two of its points:



A (hyper)plane divides spaces into two *half-spaces*. Half-spaces are convex sets! Intersection of convex sets is a convex set!



Formally a half-space is a *linear inequality*:

$$a_1x_1 + a_2x_2 + \ldots + a_dx_d \leq b$$

Definition: A polytope is a bounded subset of Euclidean space that results as the intersection of finitely many half-spaces.

An algebraic formulation for polytopes

A polytope has also an algebraic representation as the set of solutions of a system of linear inequalities:

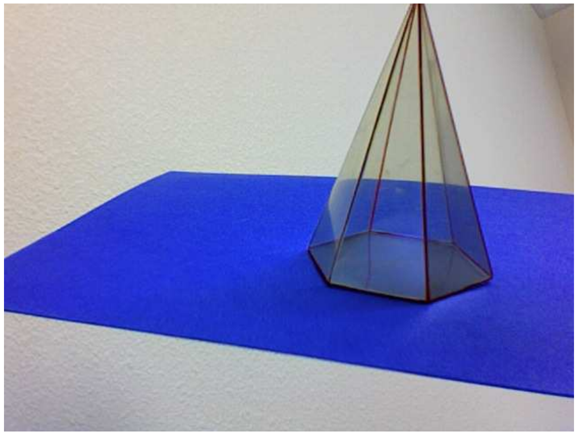
$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,d}x_d &\leq b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,d}x_d &\leq b_2 \\
&\vdots \\
a_{k,1}x_1 + a_{k,2}x_2 + \ldots + a_{k,d}x_d &\leq b_k
\end{aligned}
$$

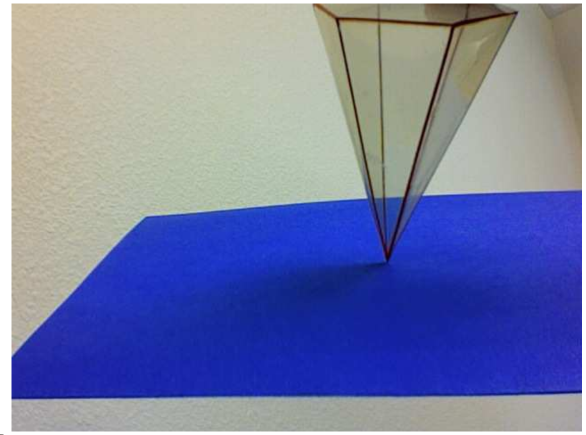
Note: This includes the possibility of using some linear equalities as well as inequalities!!

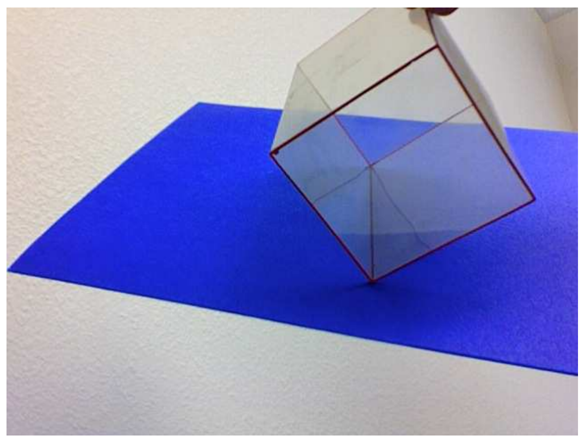
Problem 1:

NUMBERS OF FACES

Faces of Polytopes







Some Numeric Properties of Polyhedra

- **Euler's formula** $V - E + F = 2$, relates the number of vertices V, edges E, and facets F of a 3-dimensional polytope.

Given a convex 3-polytope P, if $f_i(P)$ the number of i-dimensional faces. There is one vector $(f_0(P), f_1(P), f_2(P))$. that counts faces, the f-**vector** of P.

- **Theorem** (Steinitz 1906) A vector of non-negative integers $(f_0(P), f_1(P), f_2(P)) \in \mathbb{Z}^3$ is a the f-vector of a 3-dimensional polytope if and only if

  1. $f_0(P) - f_1(P) + f_2(P) = 2$
  2. $2f_1(P) \geq 3f_0(P)$
  3. $2f_1(P) \geq 3f_2(P)$

- OPEN PROBLEM 1: Can one find similar conditions characterizing f-vectors of 4-dimensional polytopes?

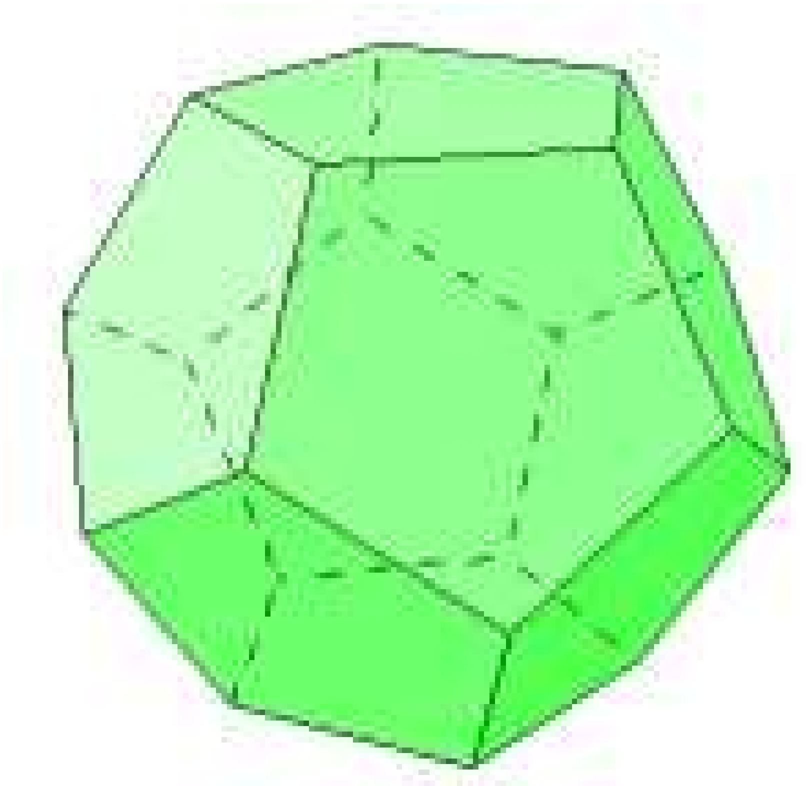
  In this case the vectors have 4 components (f_0, f_1, f_2, f_3).

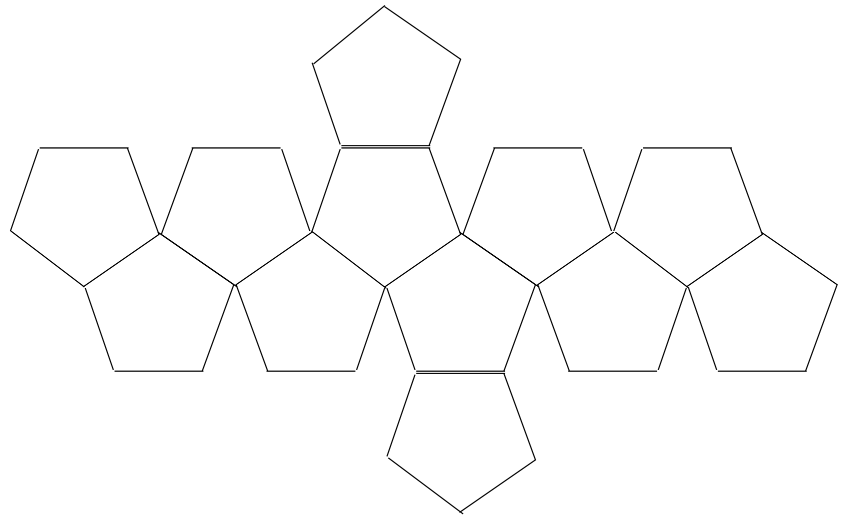
Problem 2:

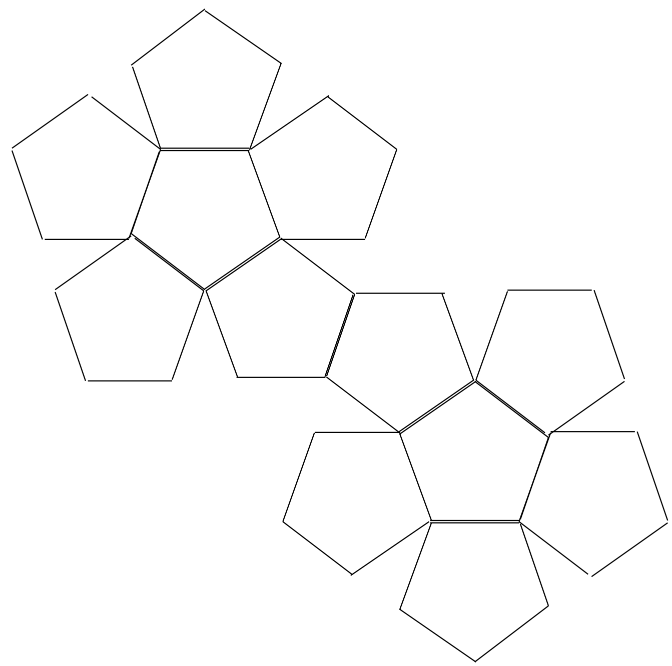
UNFOLDING POLYTOPES

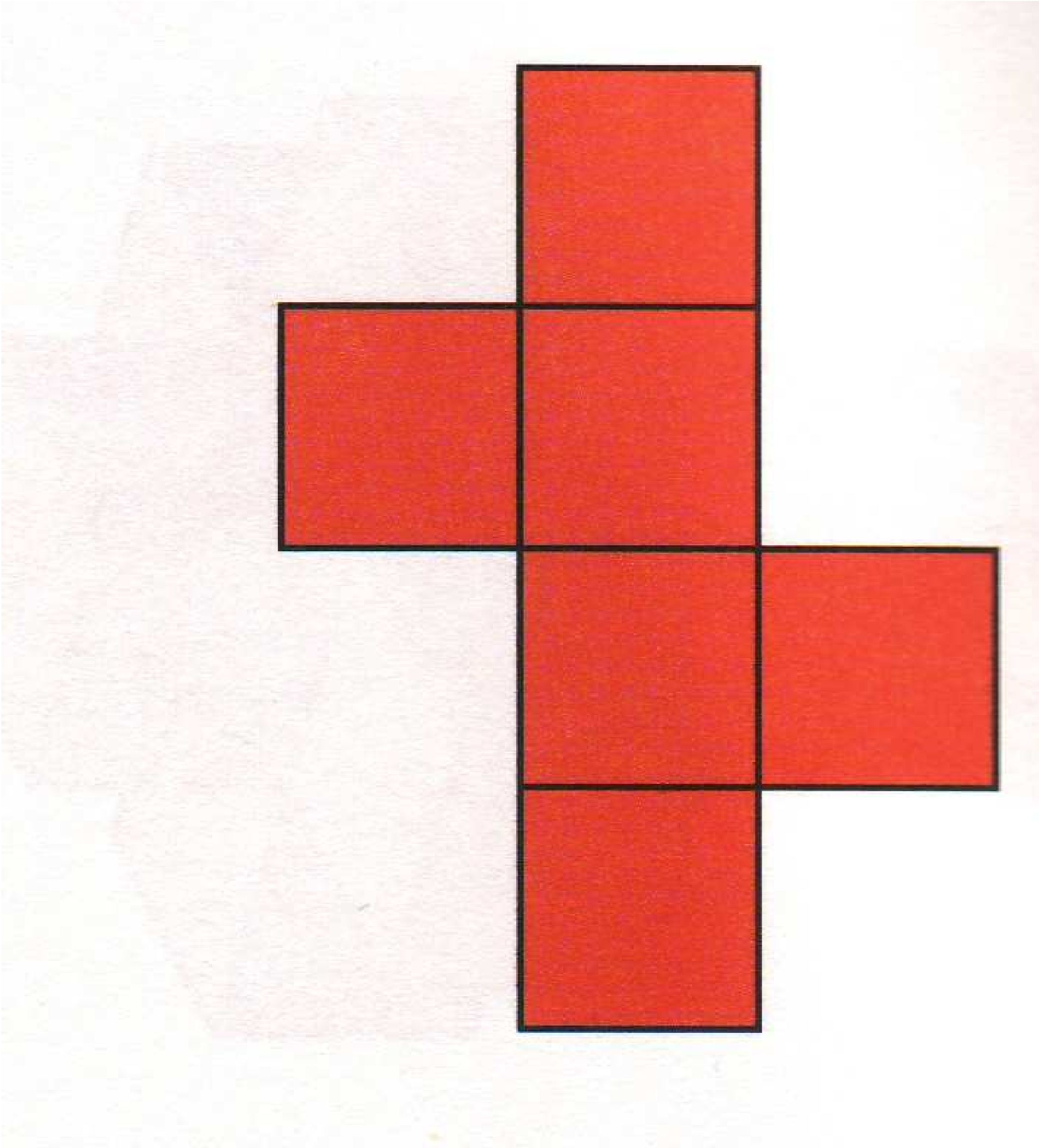
Unfolding Polyhedra

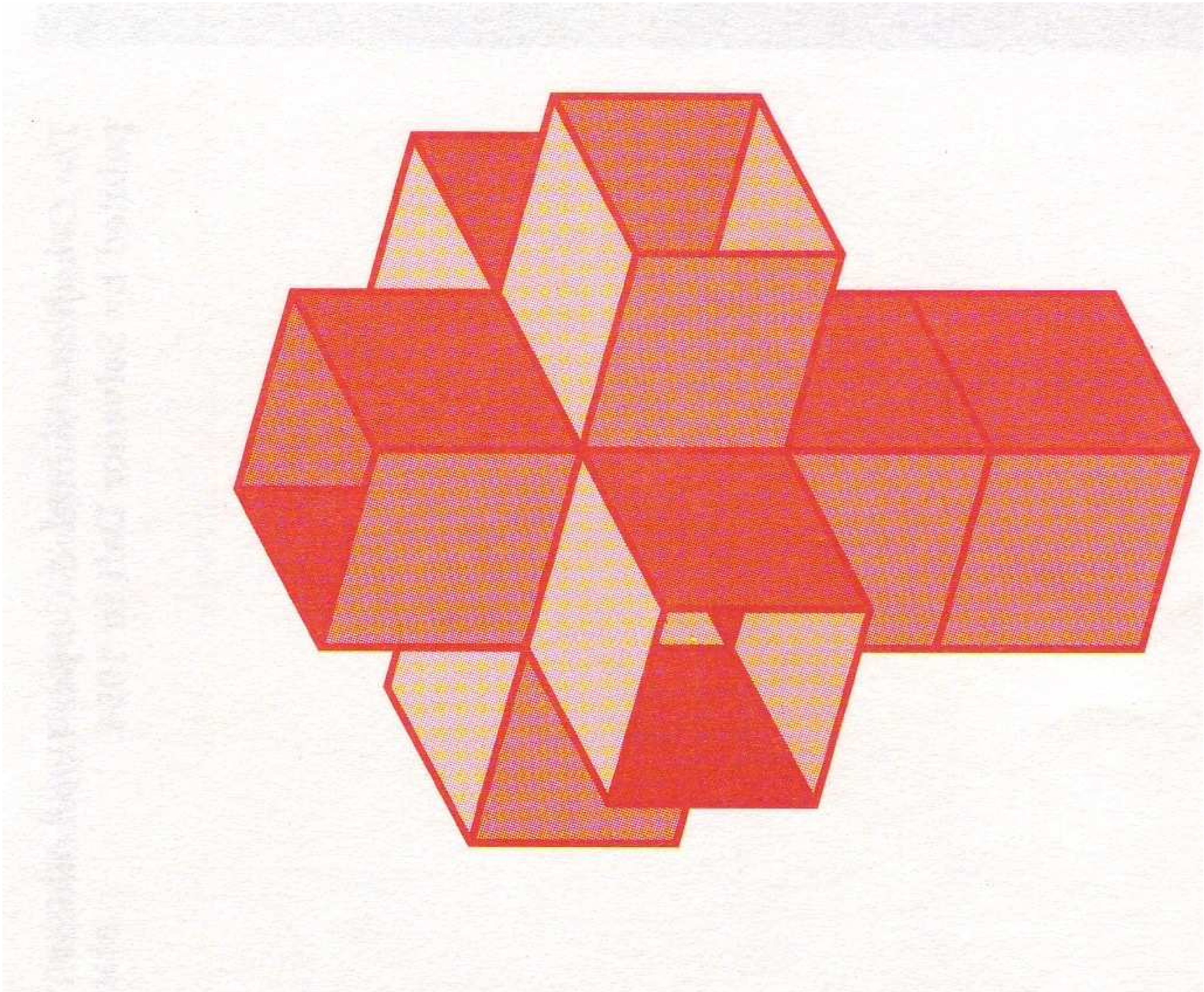
What happens if we use scissors and cut along the edges of a polyhedron? What happens to a dodecahedron?

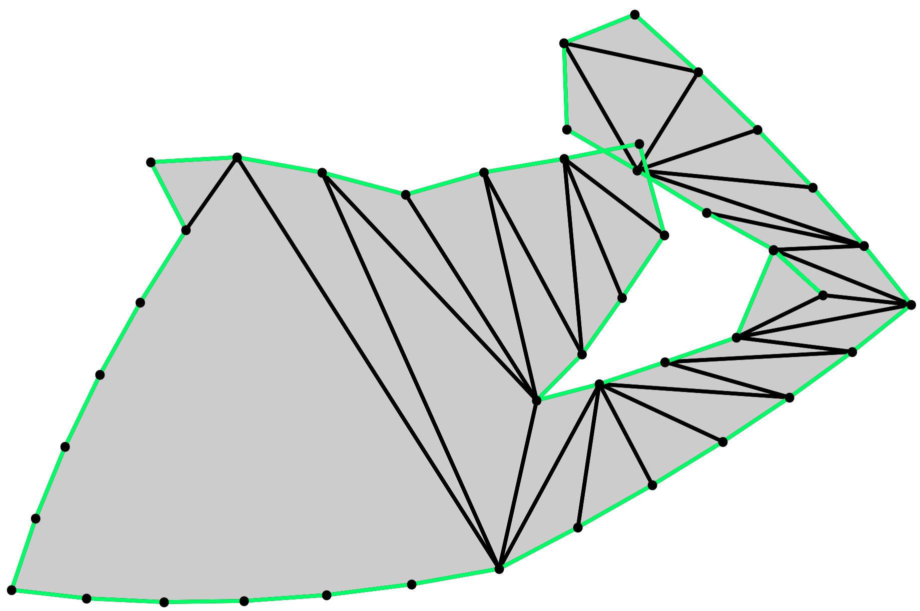


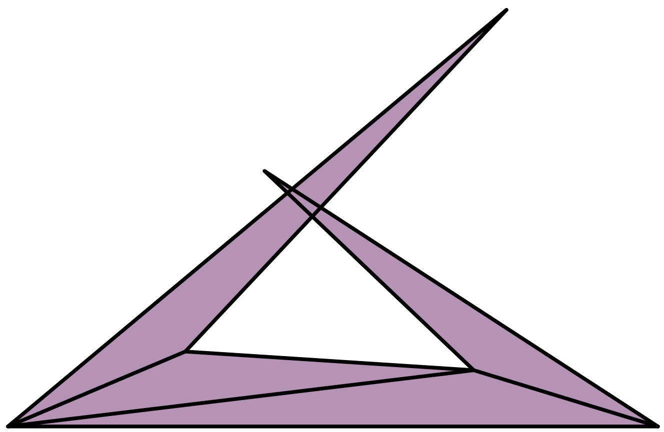












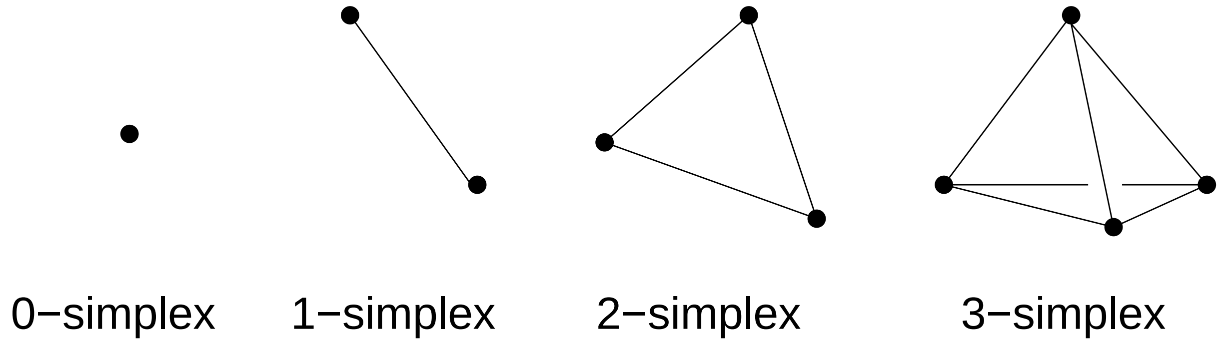
Open Problem 2: Can one always find an unfolding that has no self-overlappings for all 3-polytopes?

Problem 3:

TRIANGULATIONS

Simplices

A **simplex** is any polytope of dimension d with $d+1$ vertices.



A d-simplex has exactly $\binom{d+1}{i+1}$ faces of dimension i, ($i = -1, 0, \ldots, d$), which are themselves i-simplices.

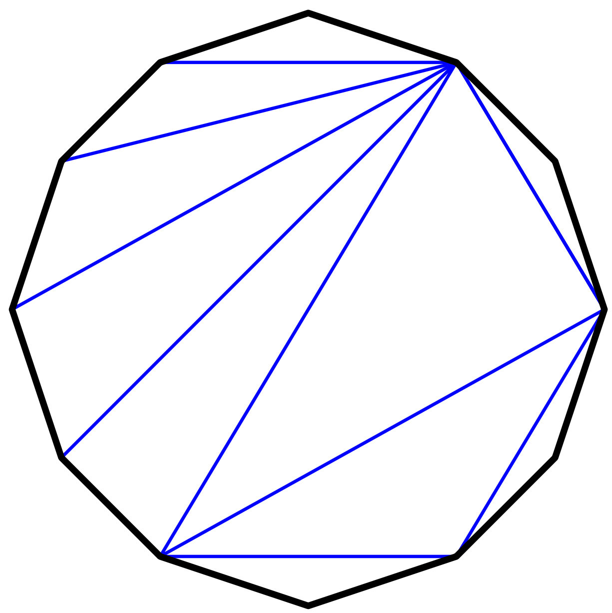
IMPORTANT: Every polytope can be decomposed as a union of simplices.

Triangulations

A **triangulation** of the polytope P is a partition of P into **simplices** such that

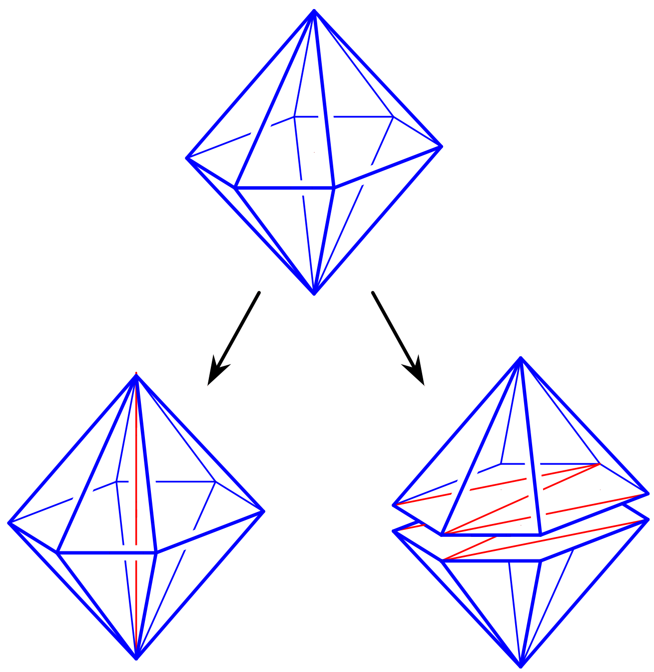
(1) The union of all these simplices equals the polytope.

(2) Any pair of simplices intersects in a (possibly empty) common face.



The size of a triangulation

Triangulations of d-polytopes come in different sizes!!



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Open Problem 3: If a polytope has triangulations of sizes k_1, k_2, with $k_2 > k_1$. Does it have a triangulation of every size S, with $k_1 < S < k_2$?

Polytopes are not just pretty faces!!

Polytopes are VERY useful!

Problem 4:

THE SIMPLEX METHOD

Linear Programming: Optimal decisions!!!

Optimal decisions by solving **Linear Programming Problems:**

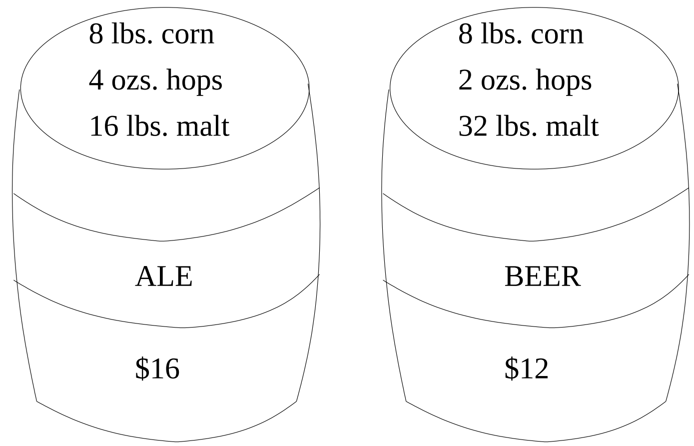
maximize $C_1 x_1 + C_2 x_2 + \ldots + C_d x_d$

among all $x_1, x_2, \ldots, x_d,$ satisfying:

$$
\begin{gathered}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,d}x_d \leq b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,d}x_d \leq b_2 \\
\vdots \\
a_{k,1}x_1 + a_{k,2}x_2 + \ldots + a_{k,d}x_d \leq b_k
\end{gathered}
$$

The Brewery Problem

A brewery produces barrels of ale or beer, each needs fixed amount of corn, hops, and malt:



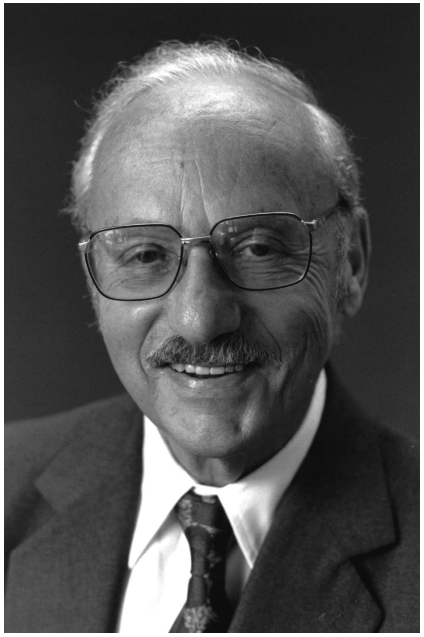
They only have 320 lbs. of corn, 130 ozs. of hops and 1120 lbs. of malt. Each barrel of ale yields a profit of $16 and each barrel of beer yields a profit of $12.

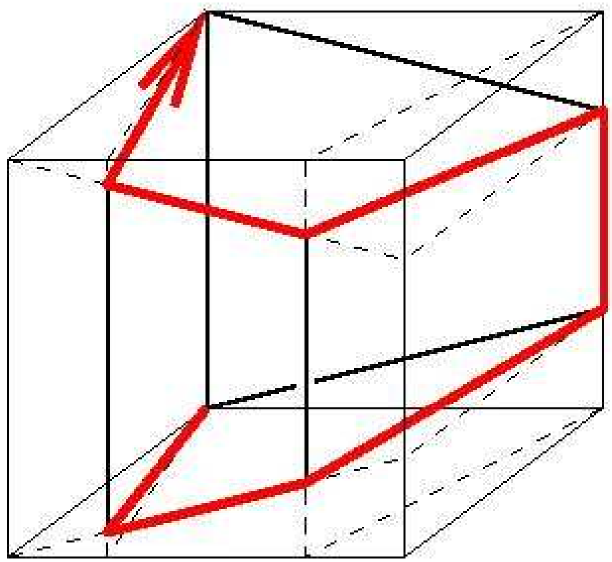
How many barrels of ale beer should be produced to maximize total profit given limited resources?

The simplex method

Lemma: A vertex of the polytope is always an optimal solution for a linear program. We need to find a special vertex of the polytope!

The simplex method **walks** along the edges of the graph of the polytope, each time moving to a better and better cost vertex!





Polynomial Hirsch Conjecture

Performance of the simplex method depends on the diameter of the graph of the polytope: largest distance between any pair of nodes.

The best UPPER bounds due to Barnette-Larman and Kalai-Kleitman are exponential:

$$(\#facets(P))^{log(dim(P))+1} \; and \quad \frac{2^{dim(P)-2}}{3}(\#facets(P) - dim(P) + 5/2).$$

While the best LOWER bound on the diameter is linear! Due to Francisco Santos 2010.

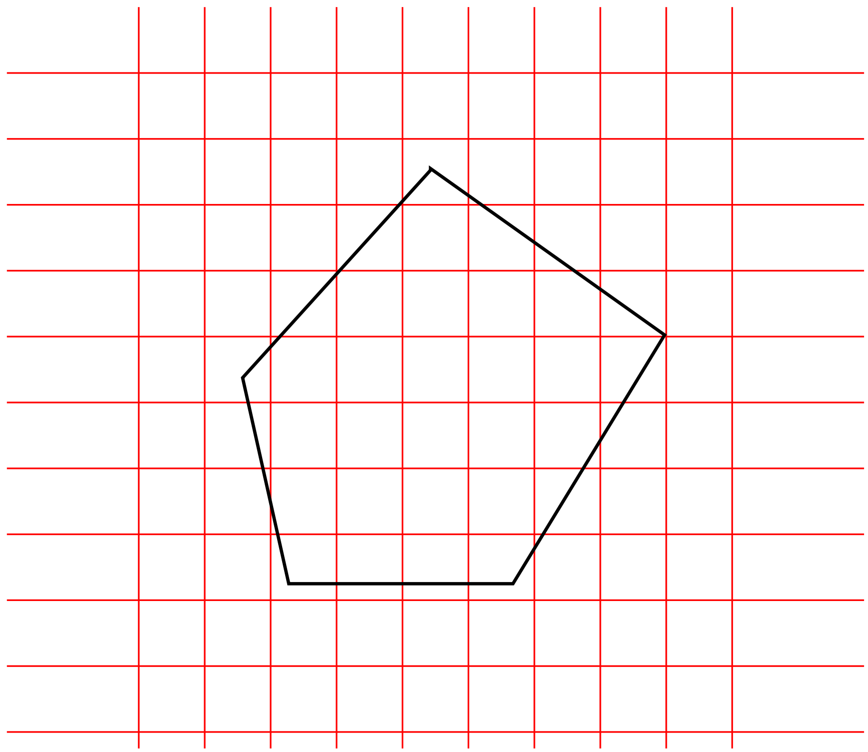
OPEN QUESTION: Is there a polynomial bound in terms of the number of facets and the dimension?

Problem 5:

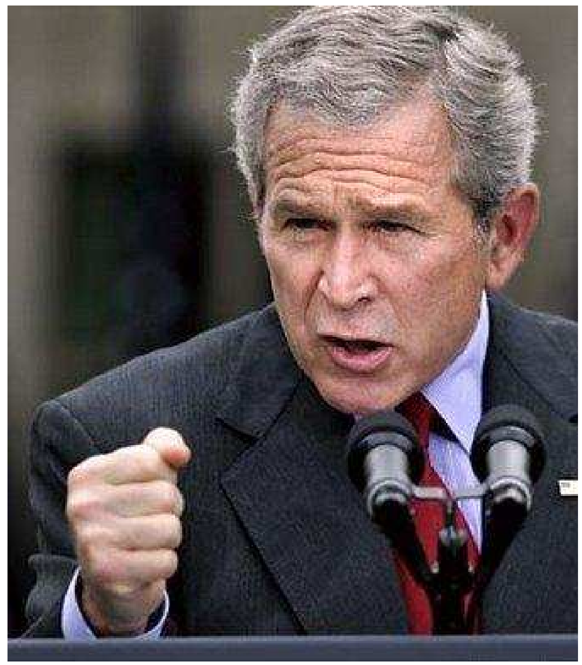
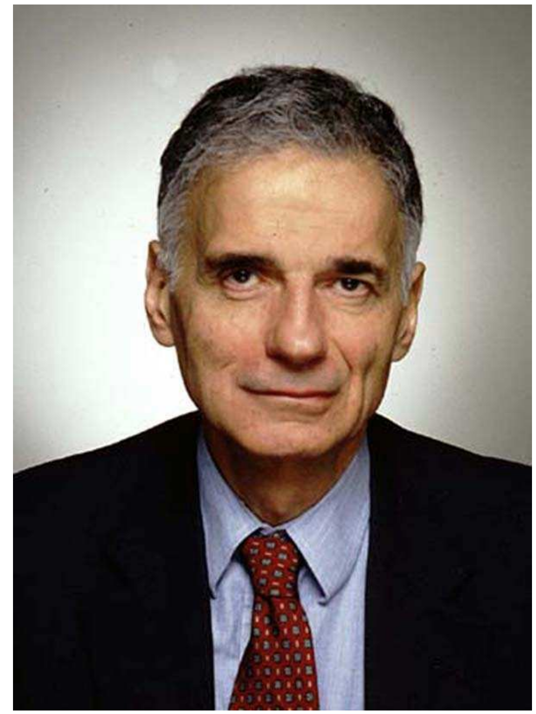
COUNTING LATTICE POINTS

Counting lattice points

Lattice points are those points with integer coordinates: $\mathbb{Z}^n = \{(x_1, x_2, \ldots, x_n) | x_i \text{ integer}\}$ We wish to count how many lie inside a given polytope!



VOTING THEORY

- There are three (presidential) candidates a, b and c....

- Let the preference orders of the $n = \sum_{i=1}^{6} n_i$ voters be

$$abc\,(n_1),\ acb\,(n_2)\ bac\,(n_3),\ bca\,(n_4),\ cab\,(n_5),\ cba\,(n_6)$$

- Here, there are n_1 voters who rank candidate a as first, b second, and c third, n_2 voters who rank b first, a second, c third, etc.

- Let us assume all 6 rankings or orderings are equally likely!

- Under **simple plurality voting**, the candidate with the most vote wins. (USA elections)

- In a **plurality runoff system**, if no candidate wins more than 50% of the vote, the two candidates with the highest vote count advance to a second voting round. (FRENCH elections)

- **Fact:** Different voting systems give different winners!!!

- Question What is the probability that the simple plurality and plurality runoff systems give different winners

- POLYTOPES are the answer:

  Set up a system of equations that describes the situation. M denotes the total number of voters.

  Say a wins by plurality but, using plurality runoff, b obtains higher score than c and a majority of voters then prefer b to a.

$$
\begin{aligned}
0 <& \quad n_1 + n_2 - n_3 - n_4 \\
0 <& \quad n_3 + n_4 - n_5 - n_6 \\
\frac{-M}{2} <& \quad -n_1 - n_2 - n_5 \\
M =& \quad n_1 + n_2 + n_3 + n_4 + n_5 + n_6 \\
0 \leq& \quad n_i, i = 1, \ldots, 6
\end{aligned}
$$

- The probability of this event equals

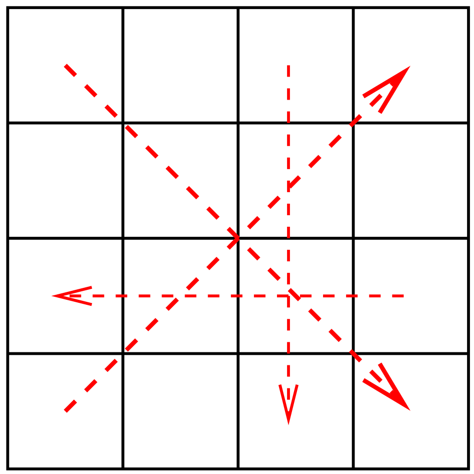
$$\frac{\textbf{Number of lattice points on polytope above}}{\textbf{number lattice pts in}\{\,(n_1,\ldots,n_6) : n_1+n_2+\ldots+n_6 = M,\ n_i \geq 0\,\}}$$

- But all 6 possible voting rankings are possible. Multiply by 6 above number, because the plurality winner may be a, b or c and the second position could be c not just b.

- ANSWER: For three candidates the probability the two voting systems give different winners for a large population is 12%.

Combinatorics via Lattice points

Many objects can be counted as the lattice points in some polytope: E.g., Sudoku configurations, matchings on graphs, and **MAGIC squares:**



12	0	5	7
0	12	7	5
7	5	0	12
5	7	12	0

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Open Problem **HOW MANY** 11×11 **magic squares with sum n are there?** Same as counting the points with integer coordinates inside the n-th dilation of a "magic square" polytope!

Indeed, we can describe it by linear constraints!

The possible magic squares are non-negative integer solutions of a system of equations and inequalities: Ten equations, one for each row sum, column sum, and diagonal sum. For example,

$x_{11} + x_{12} + x_{13} + x_{14} = 24,$ **first row**
$x_{13} + x_{23} + x_{33} + x_{43} = 24,$ **third column**, and of course $x_{ij} \geq 0$

We can do the same with tables of statistical data!!

For example, we can ask questions about gender, race, and income as done U.S. census every ten years.

Real Life Challenge: Develop fast algorithms to count how many possible tables have particular **margin statistics** for big sizes!

LIMITATION OF INFORMATION DISCLOSURE.

Gender = Male

Income Level

Race	$\leq \$10,000$	$> \$10000$ and $\leq \$25000$	$> \$25000$	Total
White	96	72	161	329
Black	10	7	6	23
Chinese	1	1	2	4
Total	107	80	169	356

Gender = Female

Income Level

Race	$\leq \$10,000$	$> \$10000$ and $\leq \$25000$	$> \$25000$	Total
White	186	127	51	364
Black	11	7	3	21
Chinese	0	1	0	1
Total	197	135	54	386

Cyber-thieves care about this!

Counting lattice points can be used to factor integer numbers of shape $n = pq$! Consider the 4-dimensional convex ball (not a polytope!):

$$B(n) = \{x \in R^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq n\}$$

Let $|B(n)| =$ the number of lattice points in $B(n)$, we have

$$|B(n)| - |B(n-1)| = 8 \sum_{m|n, m \neq 0 mod 4} m$$

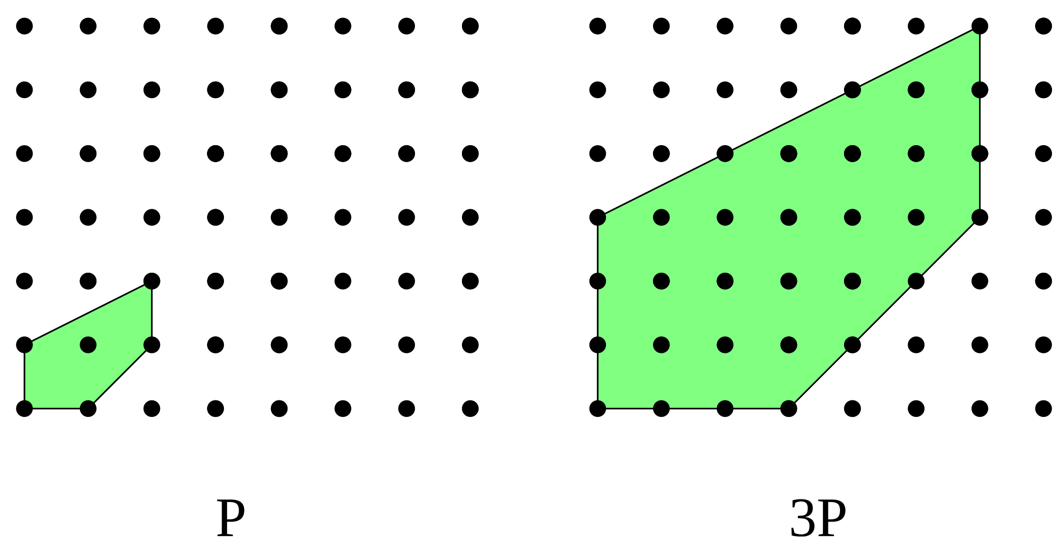
But since $n = pq$, with p, q primes, we have $|B(n)| - |B(n-1)| = 8(1 + p + q + n)$

This, together with the fact that $n = pq$ is sufficient to know p, q! **CREDIT CARDS AT RISK!**

We can approximate the volume!

Let P be a convex polytope in $\mathbb{R}^d$. For each integer $n \geq 1$, let

$$nP = \{nq | q \in P\}$$



Counting function approximates volume

For P a d-polytope, let

$$i(P,n) = \#(nP \cap \mathbb{Z}^d) = \#\{q \in P \ |nq \in \mathbb{Z}^d\}$$

This is the number of lattice points in the dilation nP.

$$\text{Volume of } P = limit_{n\to\infty} \frac{i(P,n)}{n^d}$$

At each dilation we can approximate the volume by placing a small unit cube centered at each lattice point:

Thank you! Muchas Gracias!