Knots and surfaces in 3-dimensional space

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Knots

Definition

A *knot* is a smooth embedding of the circle into 3-dimensional space.

Knots

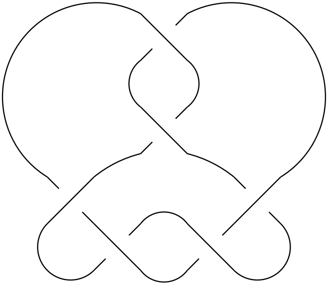

Figure: A knot

Knots

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- **Fact I:** The mathematical study of knots includes several different branches, each with a very different "flavour": algebraic, geometric, ...
- **Fact II:** Knots relate to nature, for instance via the tangles in long strands of DNA.

Definition

Given a knot K in 3-dimensional space, an orientable (2-sided) surface S that has boundary K is called a *spanning surface*.

Seifert Surfaces

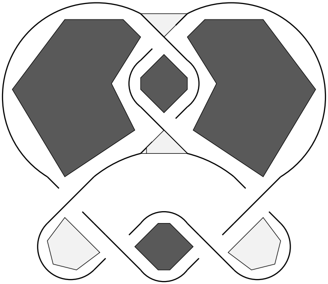

Figure: A Seifert surface

Theorem

(Seifert) Every knot admits a spanning surface.

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Seifert's algorithm

- Step 1: Consider a projection of the knot and give it an orientation.
- Step 2: Resolve each crossing to obtain Seifert circles.

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- Step 3: The Seifert circles bound disks with a natural orientation.

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Seifert's algorithm

- Step 1: Consider a projection of the knot and give it an orientation.
- Step 2: Resolve each crossing to obtain Seifert circles.
- Step 3: The Seifert circles bound disks with a natural orientation.
- Step 4: Connect disks via bands to obtain a spanning surface.

Seifert Surfaces

For a surface S constructed via Seifert's algorithm, the Euler characteristic of S, denoted by $\chi(S)$, can be computed as

$$\chi(S) = \#disks - \#bands$$

Definition

A *Seifert surface* for a knot K is a spanning surface of maximal Euler characteristic.

Seifert Surfaces (Results of Julian Eisner)

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- **Fact I:** Many knots K in $\mathbb{S}^3$ admit non-isotopic Seifert surfaces. (Eisner 1977)
- **Fact II:** Many knots K in $\mathbb{S}^3$ admit disjoint non-isotopic Seifert surfaces. (Eisner 1977)

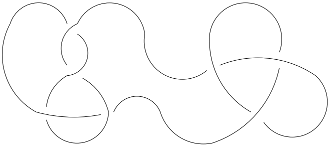

Figure: Connect sum of knots

Seifert Surfaces

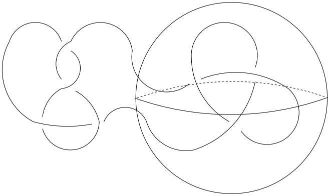

Figure: Connect sum of knots

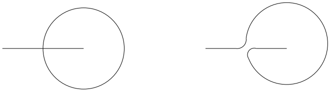

Figure: Schematic

Kakimizu complex

Definition

The vertices of the Kakimizu complex $Kak(K)$ of a knot K in $\mathbb{S}^3$ are given by the isotopy classes of minimal genus Seifert surfaces for K.

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The vertices of the Kakimizu complex $Kak(K)$ of a knot K in $\mathbb{S}^3$ are given by the isotopy classes of minimal genus Seifert surfaces for K.

The n-simplices of the Kakimizu complex of K, for $n > 1$, are given by n-tuples of vertices that admit representatives that are pairwise disjoint.

Examples of Kakimizu complexes

- **Example I:** Fibered knots have trivial Kakimizu complexes.

Examples of Kakimizu complexes

- **Example I:** Fibered knots have trivial Kakimizu complexes.
- **Example II:** Hyperbolic knots have finite Kakimizu complexes.

Structure of Kakimizu complexes

Theorem

(Scharlemann-Thompson) The Kakimizu complex of a knot is connected.

Not stated in these terms.

Theorem

(Kakimizu) Suppose that K_1, K_2 are knots with unique minimal genus Seifert surfaces (up to isotopy). Then the Kakimizu complex of $K = K_1 \# K_2$ is a bi-infinite ray.

More recently, Kakimizu computed the Kakimizu complexes for all prime knots with up to 10 crossings.

Structure of Kakimizu complexes

Theorem

(Banks) Suppose that K_1, K_2 are knots, then the Kakimizu complex of $K_1 \# K_2$ is the product of three complexes: The Kakimizu complex of K_1, the Kakimizu complex of K_2 and the complex that has underlying space $\mathbf{R}$ and vertices at the integers.

Theorem

(Banks) There exist knots with locally infinite Kakimizu complex.

Theorem

(Banks) A knot has locally infinite Kakimizu complex only if it is a satellite of either a torus knot, a cable knot or a connected sum, with winding number 0.

Jessica Banks' results

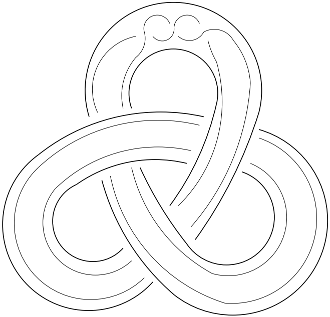

Figure: An essential torus in a knot complement

Structure of the Kakimizu complex

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Theorem

(Kapovich 2009) Let M be a Riemannian 3-manifold with smooth strictly convex boundary, together with a compact family $\mathcal{J}$ of smooth curves on ∂M. Let $f_i : (S_i, \partial S_i) \to (M, \mathcal{J})$, $i = 1, \ldots, n$ be incompressible surfaces which are pairwise non-isotopic and pairwise disjoint. Let $g_i : (S_i, \partial S_i) \to (M, \mathcal{J})$, $i = 1, \ldots, n$ be relative area minimizers in the proper isotopy classes of f_i, $i = 1, \ldots, n$. Then $g_1(S_1), \ldots, g_n(S_n)$ are also pairwise disjoint.

Contractibility of the Kakimizu complex

Theorem

(Przytycki-S 2010) The Kakimizu complex of a knot is contractible.

A projection map on the Kakimizu complex

Implicit in Kakimizu's work is a projection map (coming from considerations involving covering spaces and Kakimizu's formulation of the distance on the Kakimizu complex) that, given two vertices v, w, produces a vertex $\pi_v(w)$ that is one step closer to v than w.

$$d(v, \pi_v(w)) = d(v, w) - 1$$

Theorem

(Przytycki-S 2010) The Kakimizu complex of a knot is contractible.

Idea of proof: Choose a vertex v in $Kak(K)$ and prove that the projection map onto v is a contraction of $Kak(K)$.

Challenge: Make sure that the projection map behaves well on links of vertices.

Connectedness of the Kakimizu complex

Theorem

(Scharlemann-Thompson) The Kakimizu complex of a knot is connected.

Idea of new proof: Given any two vertices v, w, we construct a path